DIRECTED ORDERED ACYCLIC GRAPHS

ASYMPTOTIC ANALYSIS AND EFFICIENT RANDOM SAMPLING

Martin Pépin
joint work with Antoine Genitrini & Alfredo Viola

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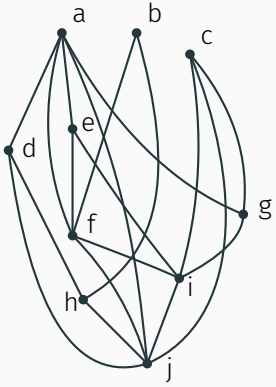
IRIF — Séminaire combi

LIPN

Directed Acyclic Graphs

Directed Acyclic Graph (DAG)

- A finite set of vertices V e.g. $\{a, b, c, \ldots, j\}$;
- a set of directed edges $E \subseteq V \times V$;
- no cycles.



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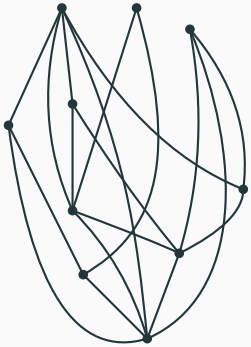

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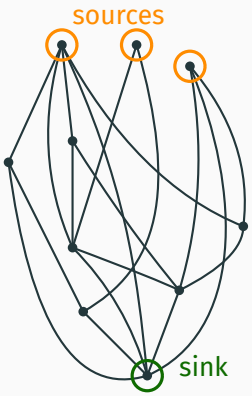

Why DAGs?

Omnipresent data structure:

- Encoding partial orders in scheduling problems;
- Git histories;
- Bayesian networks in probabilities;
- genealogy trees (those are not trees!);
- class inheritance in OOP;
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Labelled DAGs:

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Still missing

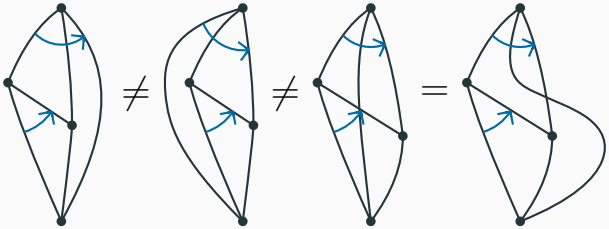
- Finer control over the number of edges?
- What can we say about compacted structures?

Outline of the presentation

A new kind of DAG

Directed Ordered Acyclic Graphs

DOAG = Unlabelled DAG
+ a total order on the **outgoing** edges of each vertex
+ a total order on the sources



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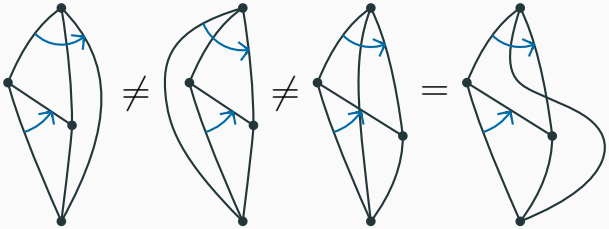
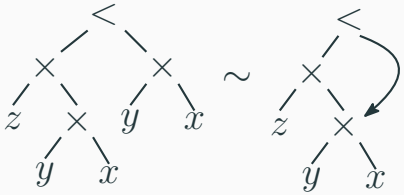
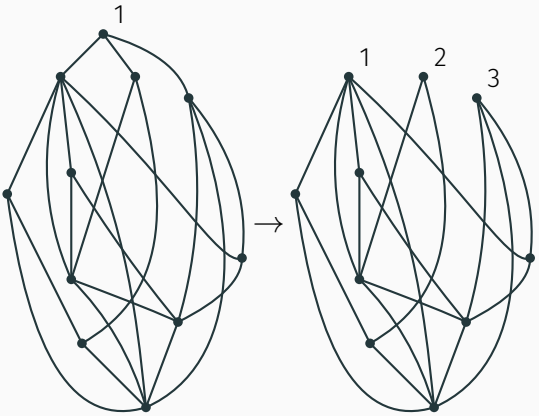




Recursive decomposition

Idea: remove the smallest source and see what is left.



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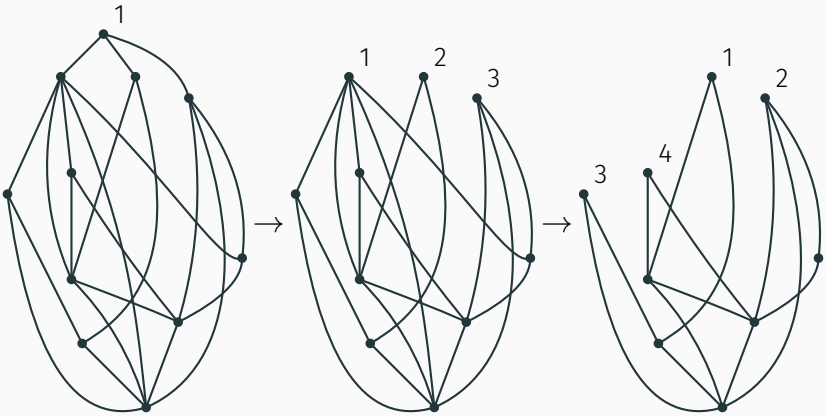

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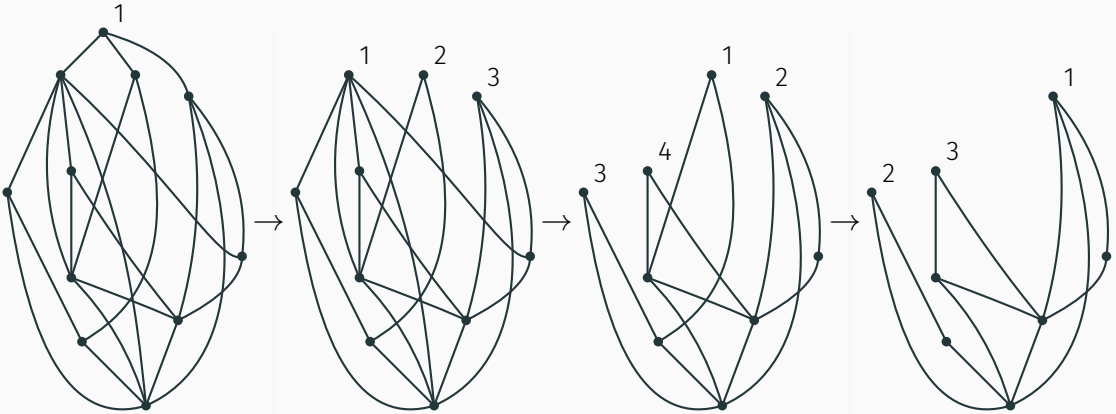

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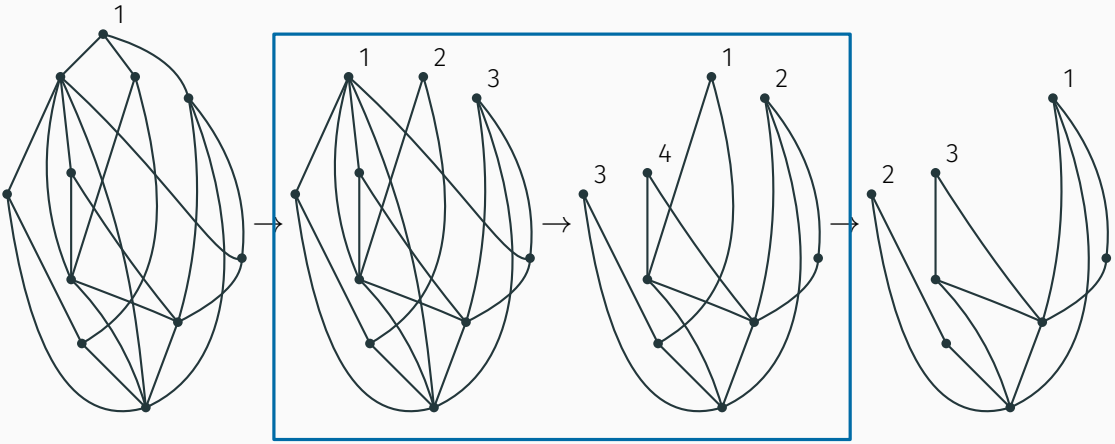

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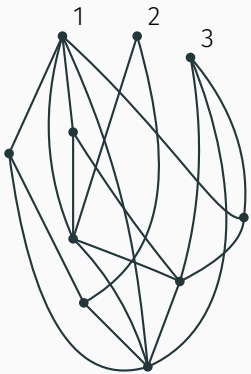

n vertices, m edges, k sources

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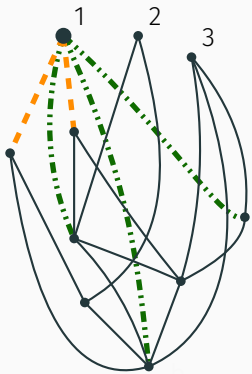

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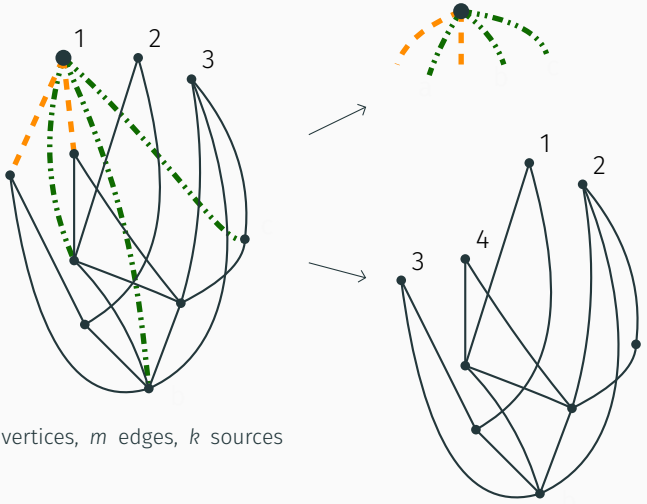

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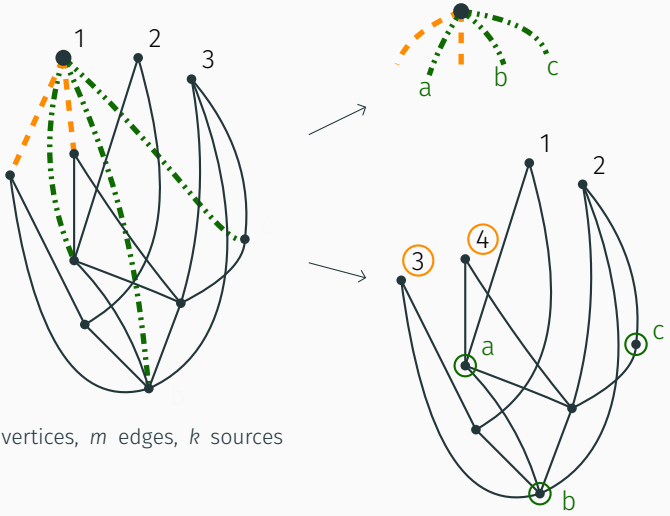

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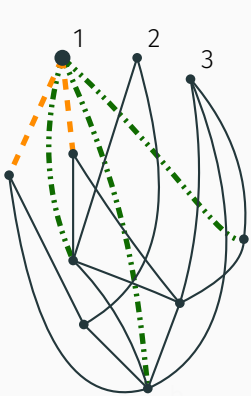
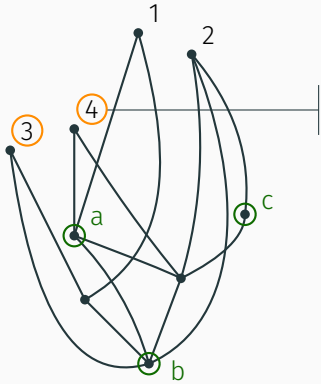


n vertices, *m* edges, *k* sources



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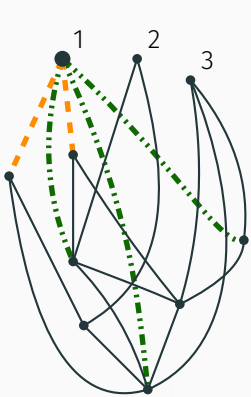

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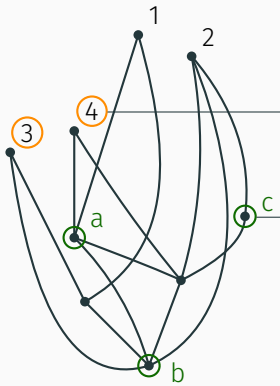

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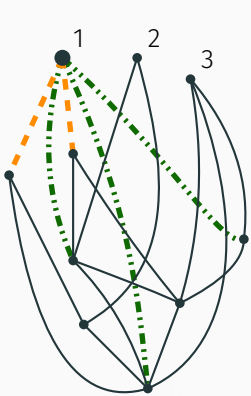

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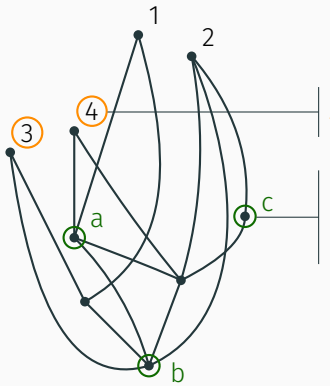

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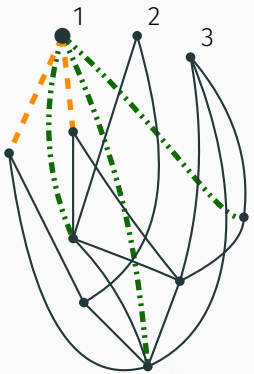

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permute the edges
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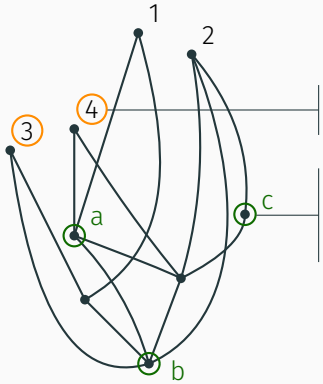

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Recurrence formula

Counting formula

$$
\begin{aligned}
D_{n,m,k} &= \#\{\textit{DOAGs with } n \textit{ vertices, } m \textit{ edges and } k \textit{ sources}\} \\
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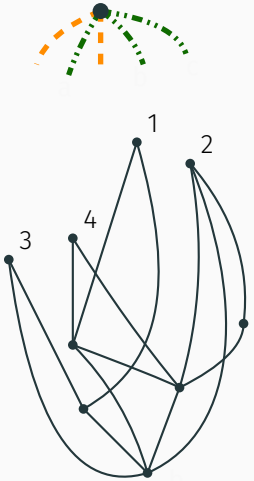

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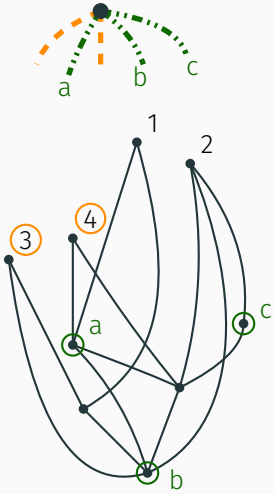

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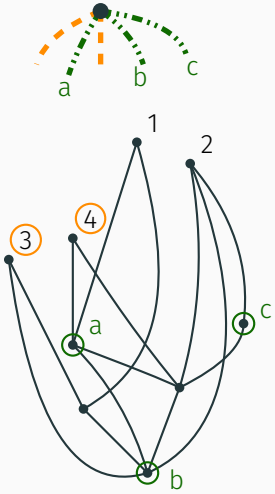

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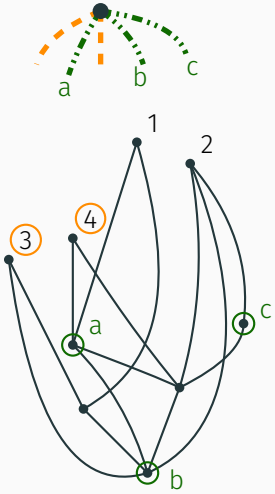

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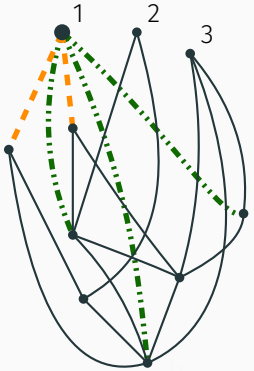
Random sampling = counting



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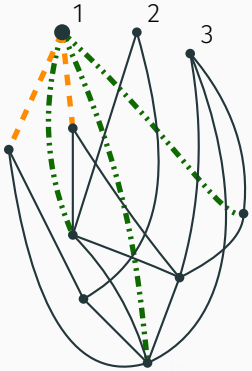

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↳ out-degree of v

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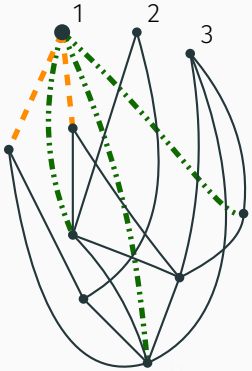

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Conclusion

- New model
- New way of counting
- Control over the number of edges

Antoine Genitrini, Martin Pépin, and Alfredo Viola. “Unlabelled ordered DAGs and labelled DAGs: constructive enumeration and uniform random sampling”. In: *XI Latin and American Algorithms, Graphs and Optimization Symposium*. Eslevier. 2021. DOI: 10.1016/j.procs.2021.11.057

Outline of the presentation

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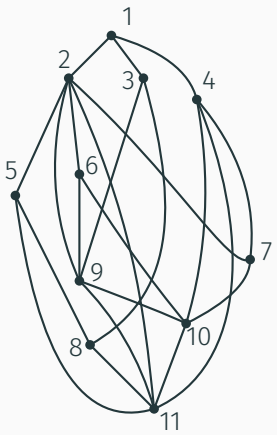
Number of single-source DOAGs

$$D_n \underset{n\to\infty}{\sim} c \cdot n^{-1/2} \cdot e^{n-1} \cdot ¡n-1!$$

for $c \approx 0.30256$ *and where* $¡x! = \prod_{k=1}^{x} k!$.

Matrix encoding



	1	2	3	4	5	6	7	8	9	10	11
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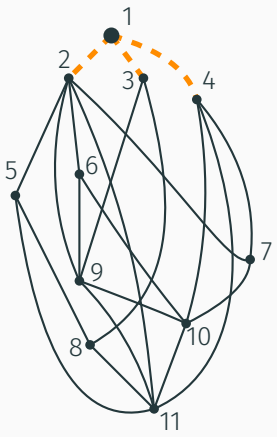

	1	2	3	4	5	6	7	8	9	10	11
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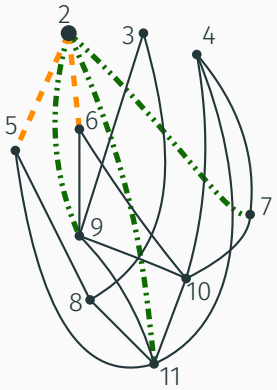
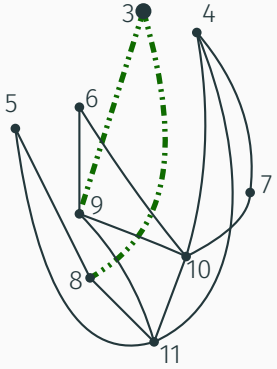
11											

Matrix encoding



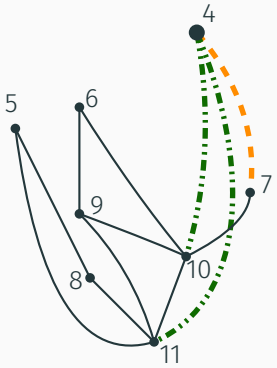
	1	2	3	4	5	6	7	8	9	10	11
1		**1**	**2**	**3**							
2					**1**	**3**	**5**		**2**		**4**
3											
4											
5											
6											
7											
8											
9											
10											
11											

Matrix encoding



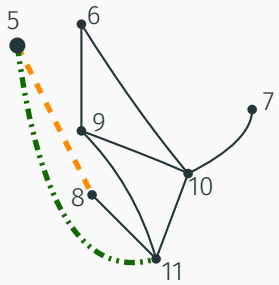
	1	2	3	4	5	6	7	8	9	10	11
1		**1**	**2**	**3**							
2					**1**	**3**	**5**		**2**		**4**
3								**2**	**1**		
4											
5											
6											
7											
8											
9											
10											
11											

Matrix encoding



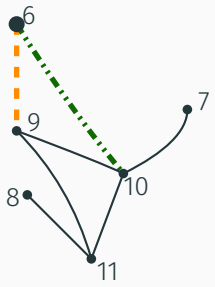
	1	2	3	4	5	6	7	8	9	10	11
1		**1**	**2**	**3**							
2					**1**	**3**	**5**		**2**		**4**
3								**2**	**1**		
4							**3**			**1**	**2**
5											
6											
7											
8											
9											
10											
11											

Matrix encoding



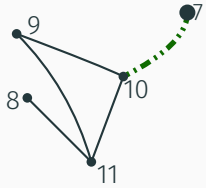
	1	2	3	4	5	6	7	8	9	10	11
1		**1**	**2**	**3**							
2					**1**	**3**	**5**		**2**		**4**
3								**2**	**1**		
4							**3**			**1**	**2**
5								**2**			**1**
6											
7											
8											
9											
10											
11											

Matrix encoding



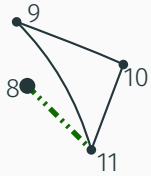
	1	2	3	4	5	6	7	8	9	10	11
1		**1**	**2**	**3**							
2					**1**	**3**	**5**		**2**		**4**
3								**2**	**1**		
4							**3**			**1**	**2**
5								**2**			**1**
6									**1**	**2**	
7											
8											
9											
10											
11											

Matrix encoding



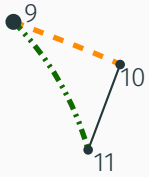
	1	2	3	4	5	6	7	8	9	10	11
1		**1**	**2**	**3**							
2					**1**	**3**	**5**		**2**		**4**
3								**2**	**1**		
4							**3**			**1**	**2**
5								**2**			**1**
6									**1**	**2**	
7										**1**	
8											
9											
10											
11											

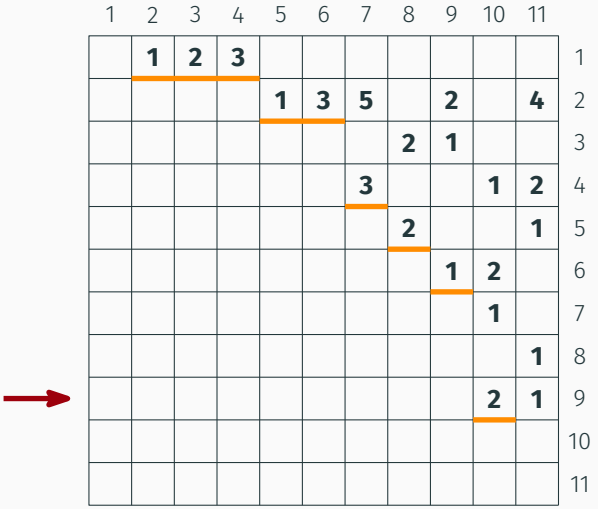
Matrix encoding



	1	2	3	4	5	6	7	8	9	10	11	
1		**1**	**2**	**3**								1
2					**1**	**3**	**5**		**2**		**4**	2
3								**2**	**1**			3
4							**3**			**1**	**2**	4
5								**2**			**1**	5
6									**1**	**2**		6
7										**1**		7
8											**1**	8
9												9
10												10
11												11

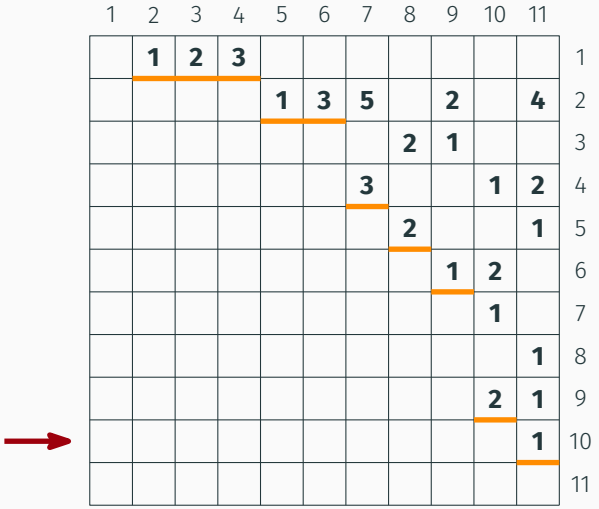
Matrix encoding



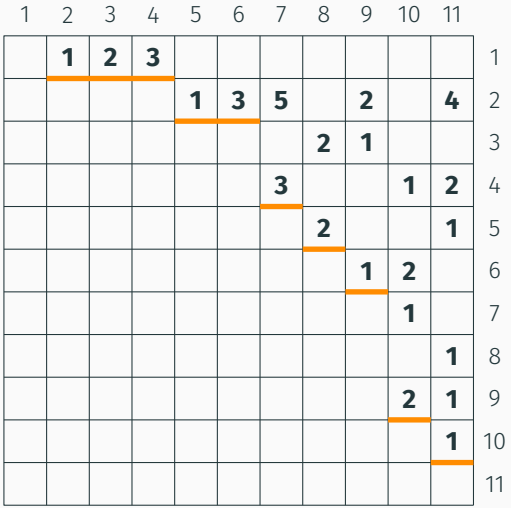


	1	2	3	4	5	6	7	8	9	10	11	
1		**1**	**2**	**3**								1
2					**1**	**3**	**5**		**2**		**4**	2
3								**2**	**1**			3
4							**3**			**1**	**2**	4
5								**2**			**1**	5
6									**1**	**2**		6
7										**1**		7
8											**1**	8
9										**2**	**1**	9
10												10
11												11

Matrix encoding

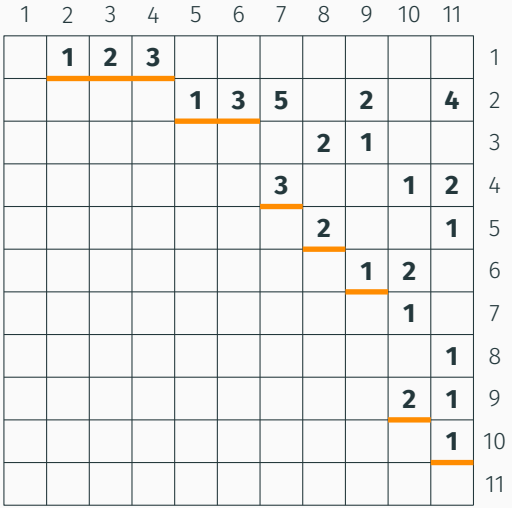


Matrix encoding



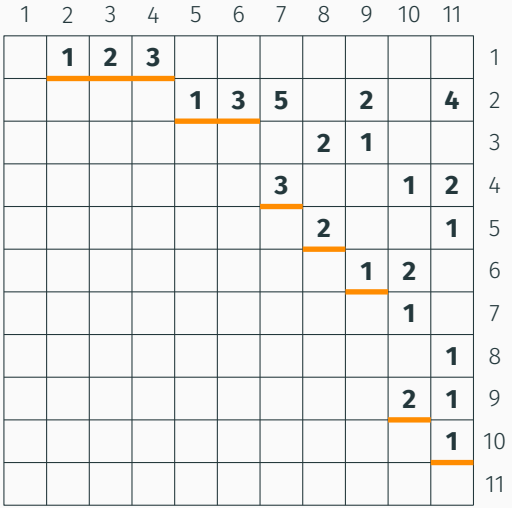
Matrix encoding

1. strict upper triangular matrix;



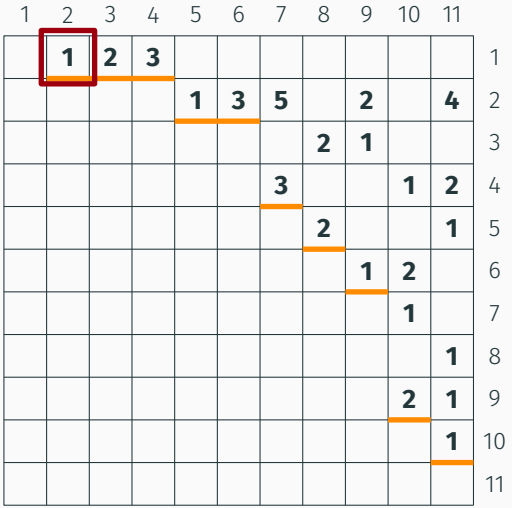
Matrix encoding

1. strict upper triangular matrix;
2. lines use an interval of values;



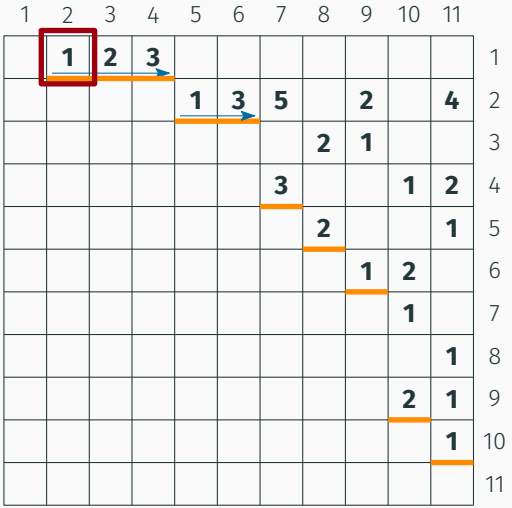
Matrix encoding

1. strict upper triangular matrix;
2. lines use an interval of values;
3. $a_{1,2} \neq 0$;



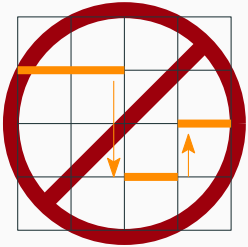
Matrix encoding

1. strict upper triangular matrix;
2. lines use an interval of values;
3. $a_{1,2} \neq 0$;
4. increasing numbers above orange lines;



Matrix encoding

1. strict upper triangular matrix;
2. lines use an interval of values;
3. $a_{1,2} \neq 0$;
4. increasing numbers above orange lines;
5. orange lines go down.



	1	2	3	4	5	6	7	8	9	10	11
1		1	2	3							
2					1	3	5		2		4
3								2	1		
4							3			1	2
5								2			1
6									1	2	
7										1	
8											1
9										2	1
10											1
11											

Proof sketch (1/3)

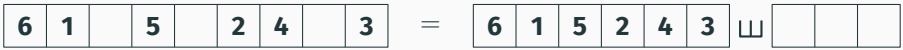
The plan: 1. Upper bound 2. Lower bound 3. Bootstrapping

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

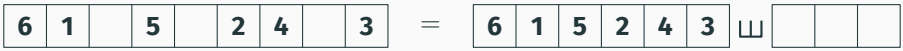
Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping



Proof sketch (1/3)

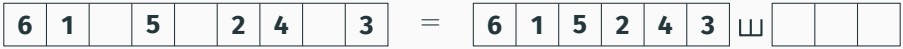
The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping



Variation $=$ $\textsc{Seq}(\mathcal{Z})$ $\star$ $\textsc{Set}(\mathcal{Z})$

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

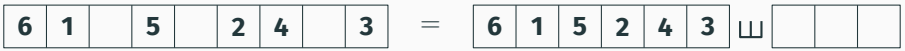


$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

$$V(z) = (1-z)^{-1} e^{z}$$

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping



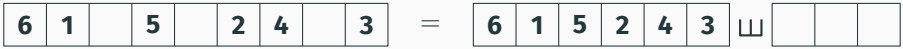
$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

$$V(z) = (1-z)^{-1} e^{z}$$

$$v_n = e \cdot n! - o(1)$$

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping



$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

$$V(z) = (1-z)^{-1}e^{z}$$

$$v_n = e \cdot n! - o(1)$$

$\#\{\text{DOAG matrices}\} = \#\{\text{collections of rows}\} \leq \#\{\text{collections of variations}\}$

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

6	1		5		2	4		3	=	6	1	5	2	4	3	⊔			

$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

$$V(z) = (1-z)^{-1}e^{z}$$

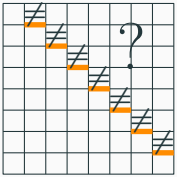
$$v_n = e \cdot n! - o(1)$$

$\#\{\text{DOAG matrices}\} = \#\{\text{collections of rows}\} \leq \#\{\text{collections of variations}\}$

$$D_n \leq \prod_{k=1}^{n-1} v_k \leq e^{n-1}\ ¡n-1!$$

Proof sketch (2/3)

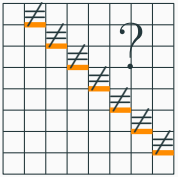
The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$ 

(constraints are automatically satisfied)

Proof sketch (2/3)

The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

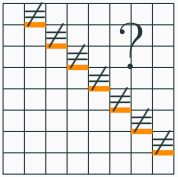
{DOAG matrices} $\supseteq$ 

(constraints are automatically satisfied)

$\#$  $= v_k - v_{k-1}$

Proof sketch (2/3)

The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

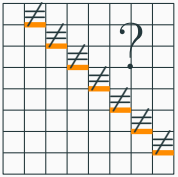
{DOAG matrices} $\supseteq$ 

(constraints are automatically satisfied)

$$\#\boxed{\neq}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?} = v_k - v_{k-1} = e \cdot k! \cdot \left(1 - \frac{1}{k} - o\left(\frac{1}{(k-1)!}\right)\right)$$

Proof sketch (2/3)

The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$ 

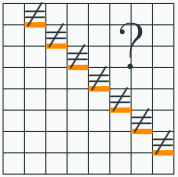
(constraints are automatically satisfied)

$$\#\boxed{\neq}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?} = v_k - v_{k-1} = e \cdot k! \cdot \left(1 - \frac{1}{k} - o\left(\frac{1}{(k-1)!}\right)\right)$$

$$D_n \geq e^{n-1}\,¡n-1!\prod_{k=2}^{n-1}\left(\frac{k-1}{k} + o\left(\frac{1}{(k-1)!}\right)\right)$$

Proof sketch (2/3)

The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$ 

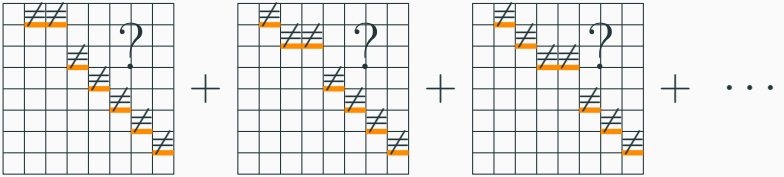
(constraints are automatically satisfied)

$$\#\boxed{\neq}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?} = v_k - v_{k-1} = e \cdot k! \cdot \left(1 - \frac{1}{k} - o\left(\frac{1}{(k-1)!}\right)\right)$$

$$D_n \geq e^{n-1}\,¡n-1!\prod_{k=2}^{n-1}\left(\frac{k-1}{k} + o\left(\frac{1}{(k-1)!}\right)\right) \geq e^{n-1}\,¡n-1!\,\frac{A}{n} \quad \text{for some } A > 0$$

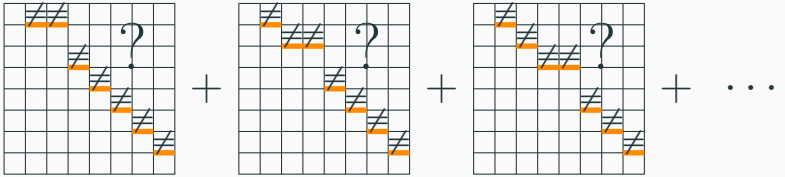
Proof sketch (2’/3)

The plan: 1. Upper bound **2’. Better lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$  + + + $\cdots$

Proof sketch (2'/3)

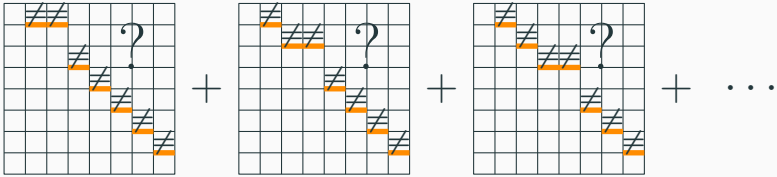
The plan: 1. Upper bound **2'. Better lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$  $+ \cdots$

$$D_n \geq \frac{A' \cdot \ln(n)}{n} \, e^{n-1} \, ¡n - 1!$$

Proof sketch (2'/3)

The plan: 1. Upper bound **2'. Better lower bound** 3. Bootstrapping

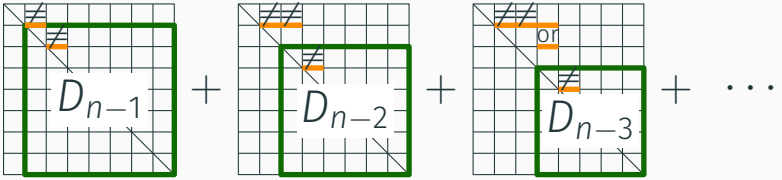
$\{\text{DOAG matrices}\} \supseteq$  $+ \cdots$

$$D_n \geq \frac{A' \cdot \ln(n)}{n} \, e^{n-1} \, ¡n-1!$$

$$P_n = \frac{D_n}{e^{n-1} \, ¡n-1!} \quad \Rightarrow \quad \frac{A' \cdot \ln(n)}{n} \leq P_n \leq 1$$

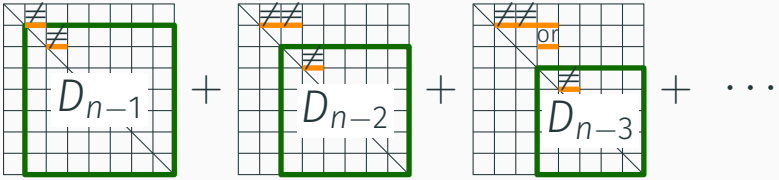
Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

{DOAG matrices} = 

Proof sketch (3/3)

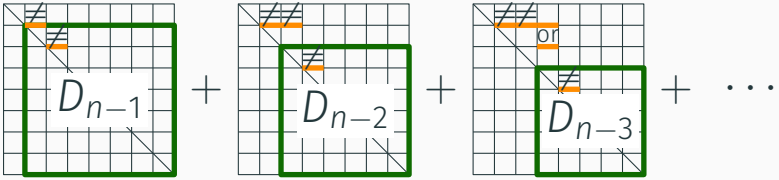
The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

$\{\text{DOAG matrices}\} =$ 

$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

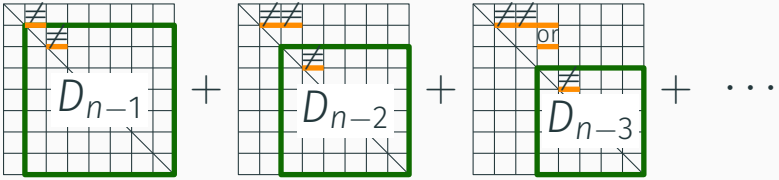
$\{\text{DOAG matrices}\} =$ 

$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

$$P_n = \left(1 - \frac{1}{n-1}\right) P_{n-1} + \frac{1}{2(n-2)} \left(1 - \frac{2}{n-1} + \frac{1}{(n-1)(n-2)}\right) P_{n-2} + \cdots$$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

$\{\text{DOAG matrices}\} =$ 

$+ \cdots$

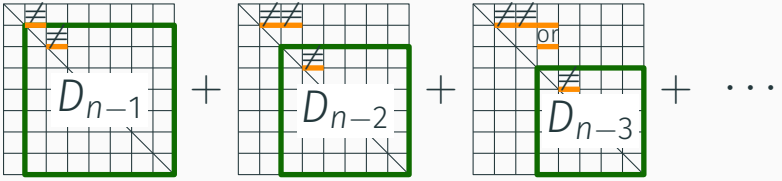
$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

$$P_n = \left(1 - \frac{1}{n-1}\right) P_{n-1} + \frac{1}{2(n-2)}\left(1 - \frac{2}{n-1} + \frac{1}{(n-1)(n-2)}\right) P_{n-2} + \cdots$$

$$P_n \sim P_{n-1}$$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

{DOAG matrices} = 

$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

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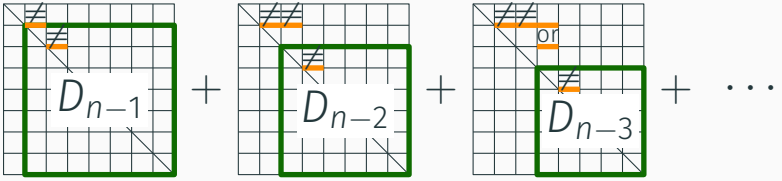
{DOAG matrices} = 

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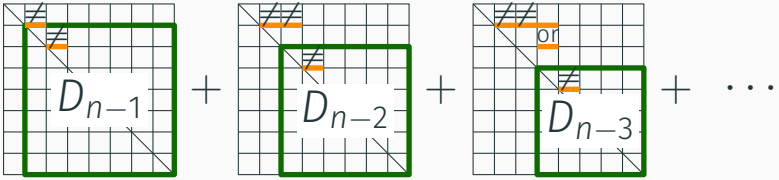
{DOAG matrices} = 

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$$P_n = P_{n-1}\left(1 - \frac{1}{2n} + O\left(n^{-2}\right)\right) \Rightarrow \boxed{P_n \sim c \cdot n^{-1/2}}$$

Random sampling again!

Corollary

$$\frac{D_n}{\#\{\textit{matrices of variations of sizes } 1, 2, \ldots, n-1\}} \sim c \cdot n^{-\frac{1}{2}}$$

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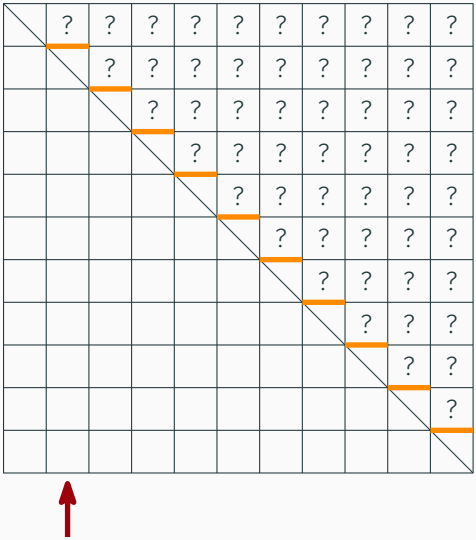
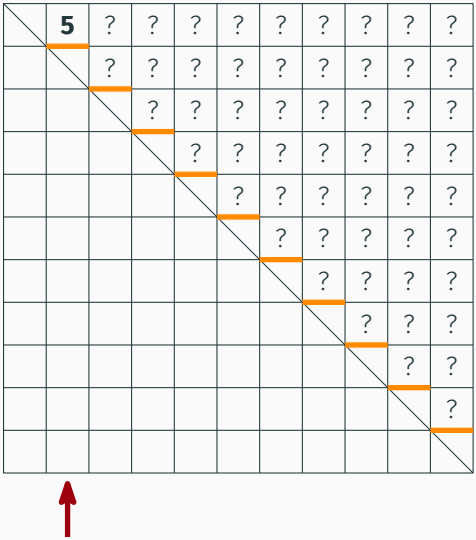
Better complexity:

$$\begin{aligned}&\text{Cost}(\text{one full generation}) + \#\text{rejections} \times \text{Cost}(\text{one failed generation})\\ &= \frac{n^2}{2}\log_2(n) + O\big(\sqrt{n} \cdot \textbf{Cost}(\text{one failed generation})\big)\end{aligned}$$

Early rejection



Early rejection



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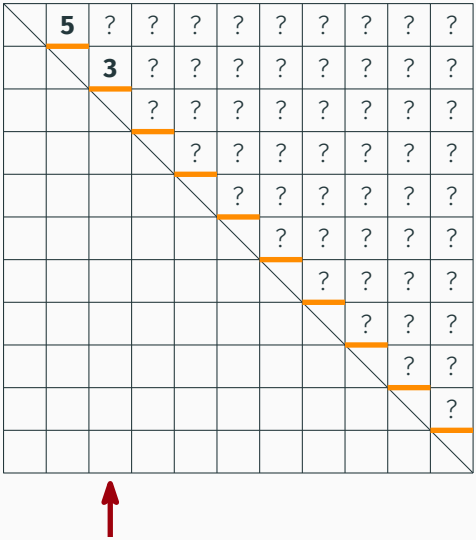

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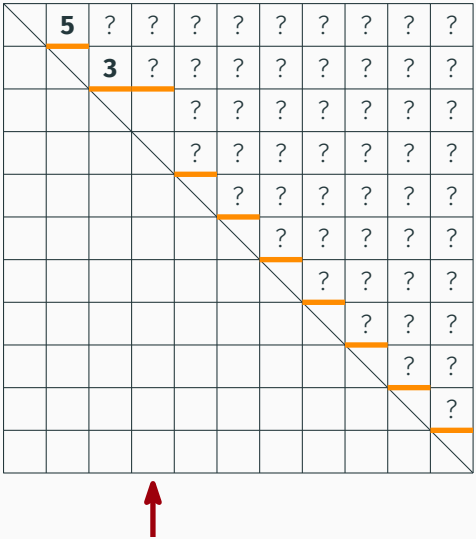

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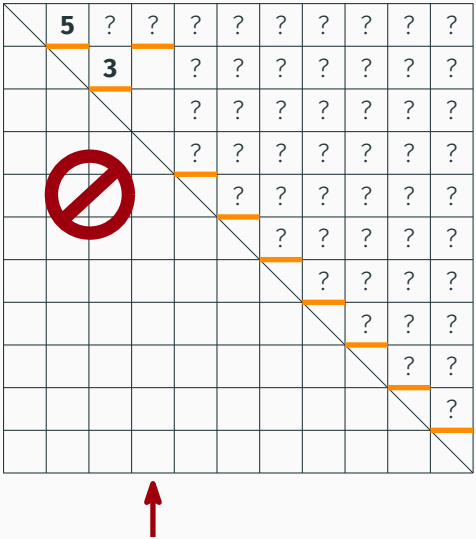

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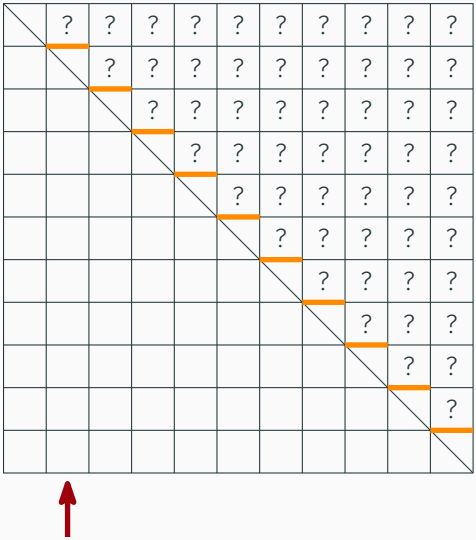

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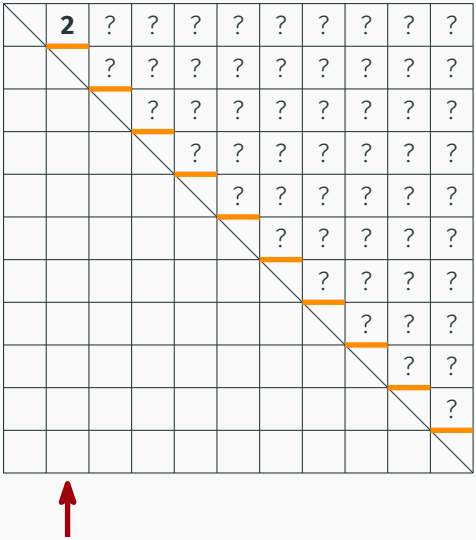

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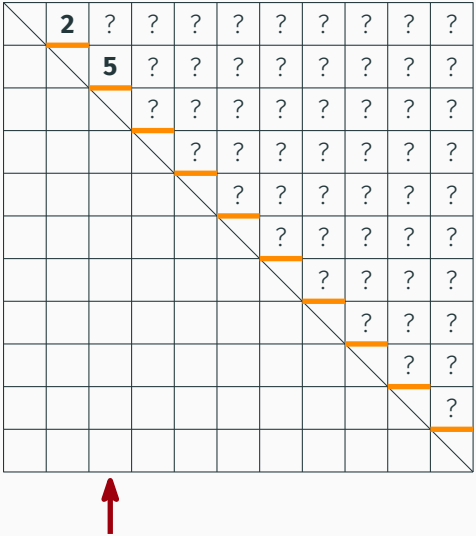

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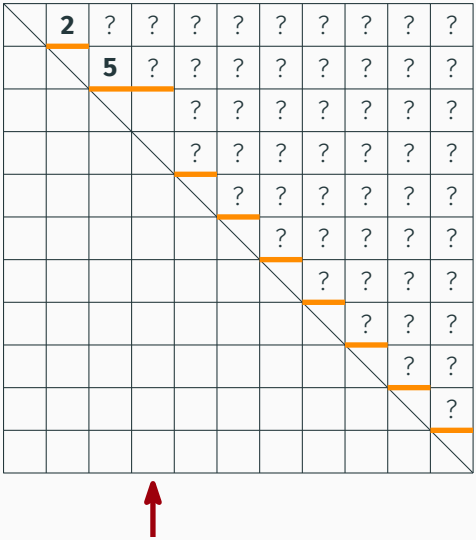

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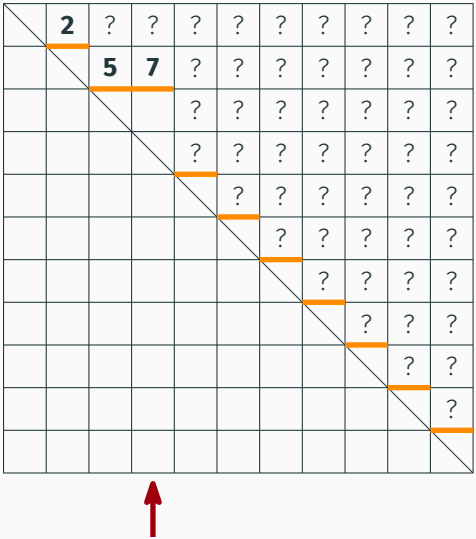

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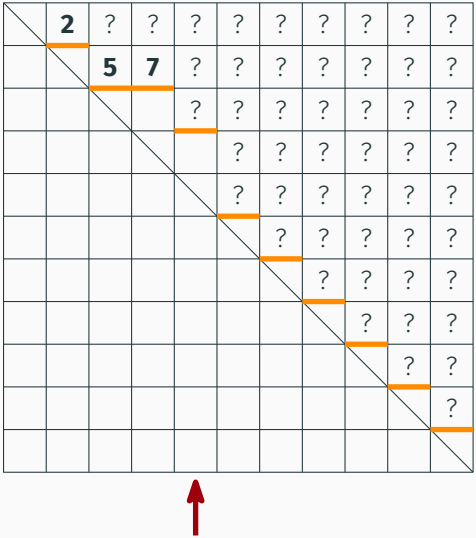

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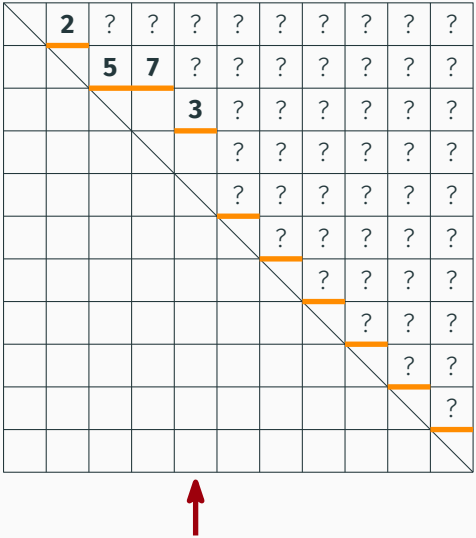

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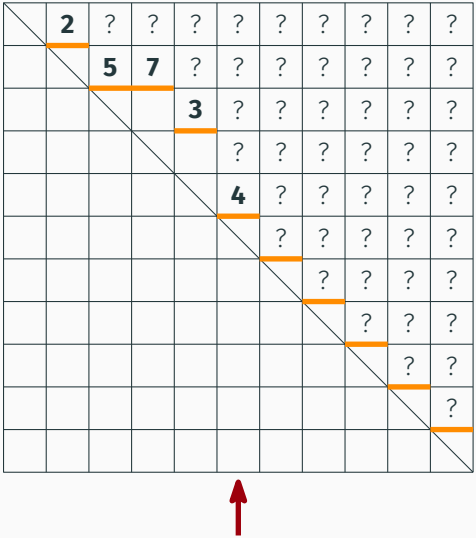

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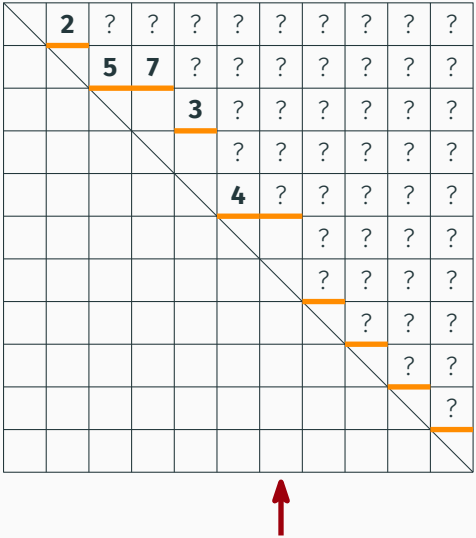

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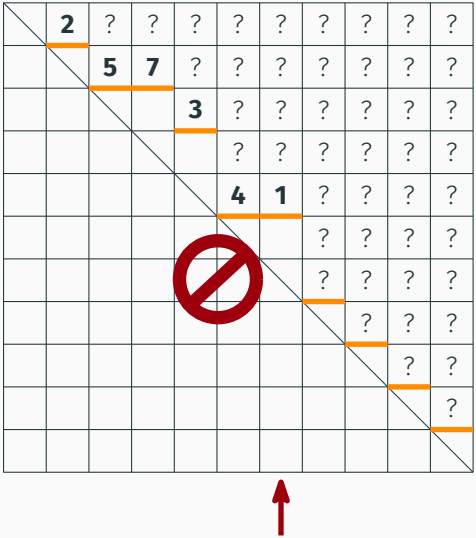

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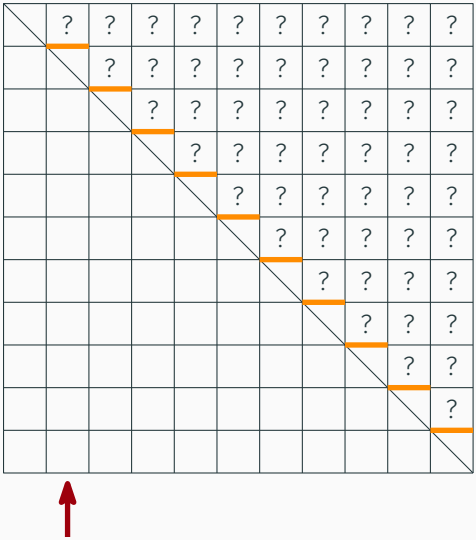

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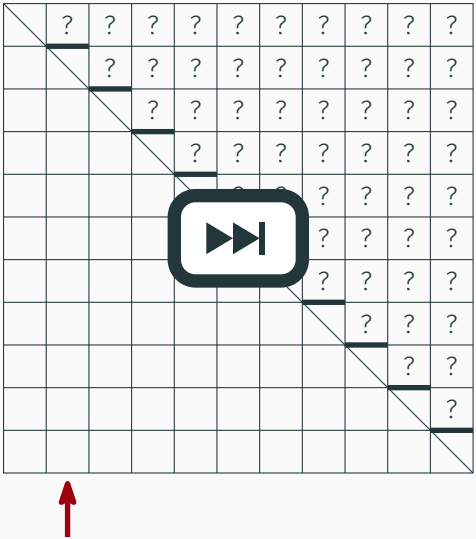

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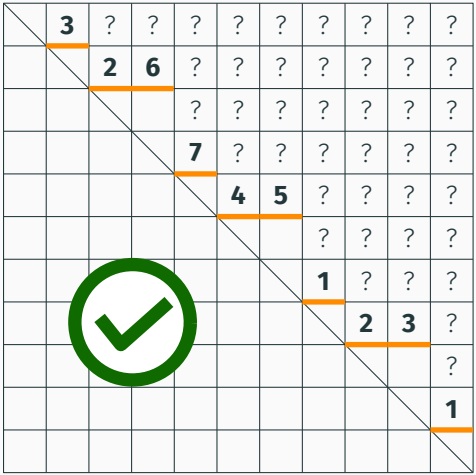

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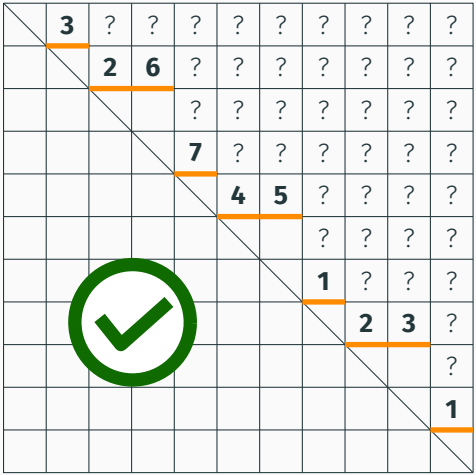

Complexity $= O(n \ln(n))$ Total complexity $= \frac{n^2}{2} \log_2(n) + O\left(\sqrt{n} \cdot n \ln(n)\right)$

Perspectives

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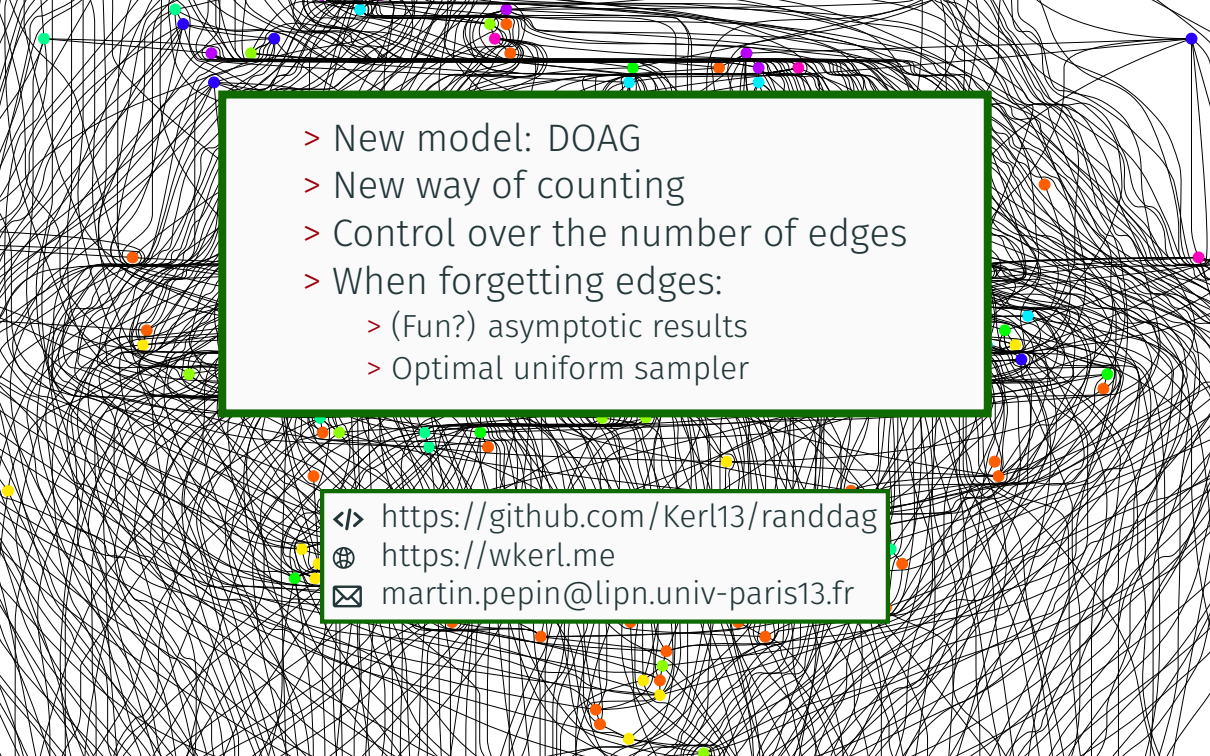
- Do something similar for labelled DAGs



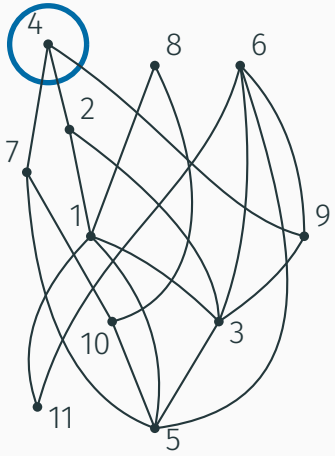
- New model: DOAG
- New way of counting
- Control over the number of edges
- When forgetting edges:
  - (Fun?) asymptotic results
  - Optimal uniform sampler

</> https://github.com/Kerl13/randdag
https://wkerl.me
martin.pepin@lipn.univ-paris13.fr

Outline of the presentation

What about labelled DAGs?

Idea: mark one source, and remove it.



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$k\,A_{n,m,k} =$

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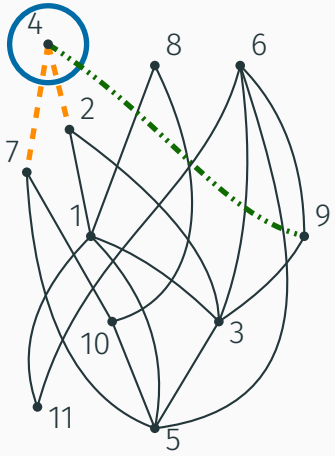

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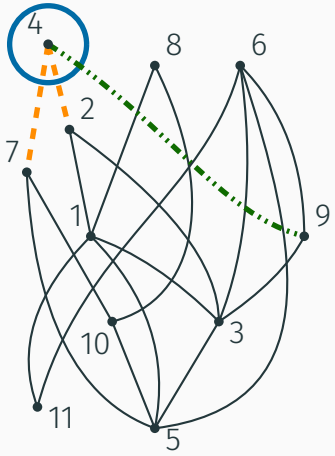

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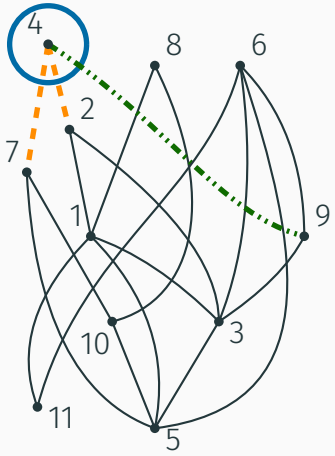

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→ New counting formula for DAGs;
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References I

[EFW21] Andrew Elvey Price, Wenjie Fang, and Michael Wallner. “Compacted binary trees admit a stretched exponential”. In: *Journal of Combinatorial Theory, Series A* 177 (2021), page 105306. ISSN: 0097-3165. DOI: 10.1016/j.jcta.2020.105306.

[Ges96] Ira Martin Gessel. “Counting acyclic digraphs by sources and sinks”. In: *Discrete Mathematics* 160.1 (1996), pages 253–258. ISSN: 0012-365X.

[GGKW20] Antoine Genitrini et al. “Asymptotic enumeration of compacted binary trees of bounded right height”. In: *Journal of Combinatorial Theory, Series A* 172 (2020), page 105177. ISSN: 0097-3165. DOI: https://doi.org/10.1016/j.jcta.2019.105177.

[GPV21] Antoine Genitrini, Martin Pépin, and Alfredo Viola. “Unlabelled ordered DAGs and labelled DAGs: constructive enumeration and uniform random sampling”. In: *XI Latin and American Algorithms, Graphs and Optimization Symposium*. Eslevier. 2021. DOI: 10.1016/j.procs.2021.11.057.

[KM15] Jack Kuipers and Giusi Moffa. “Uniform random generation of large acyclic digraphs”. In: *Statistics and Computing* 25.2 (2015), pages 227–242.

References II

[MDB01] Guy Melançon, Isabelle Dutour, and Mireille Bousquet-Mélou. “Random Generation of Directed Acyclic Graphs”. In: *Electronic Notes in Discrete Mathematics* 10 (2001), pages 202–207. DOI: 10.1016/S1571-0653(04)00394-4. URL: https://doi.org/10.1016/S1571-0653(04)00394-4.

[Rob70] Robert William Robinson. “Enumeration of acyclic digraphs”. In: *Proceedings of The Second Chapel Hill Conference on Combinatorial Mathematics and its Applications (Univ. North Carolina, Chapel Hill, NC, 1970), Univ. North Carolina, Chapel Hill, NC* (University of North Carolina at Chapel Hill, North Carolina, May 8–13, 1970). 1970, pages 391–399.

[Rob73] Robert William Robinson. “Counting labeled acyclic digraphs”. In: *New Directions in the Theory of Graphs* (1973), pages 239–273.

[Rob77] Robert William Robinson. “Counting unlabeled acyclic digraphs”. In: *Combinatorial Mathematics V*. Lecture Notes in Mathematics. Springer, 1977, pages 28–43.

[Sta73] Richard Peter Stanley. “Acyclic orientations of graphs”. In: *Discrete Mathematics* 5.2 (1973), pages 171–178.