Statistical Analysis of Non-Deterministic Fork-Join Processes

> Martin Pépin Antoine Genitrini Frédéric Peschanski

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Concurrency

One computation unit shared by several processes:

- Possible dependencies between processes
- Scheduling

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How to ensure that a program is correct regardless of the **scheduling**?

- Many possible schedulings: combinatorial explosion.
- Can we (efficiently) count them?
- Can we (efficiently) sample uniformly among them?

A negative result

[Brightwell & Winkler '91]

Counting the linear extensions of a partial order is a #-*P* complete problem.

I.e. it is as hard as counting the number of solutions in SAT.

So we cannot count efficiently...

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Counting the linear extensions of a partial order is a #-*P* complete problem.

I.e. it is as hard as counting the number of solutions in SAT.

So we cannot count efficiently... in the **general** case.
But we can have some restrictions on the programs.

The long-term project

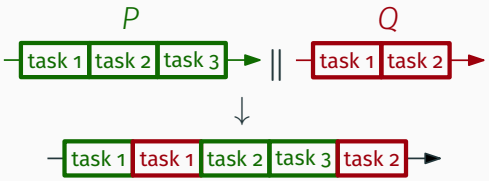
"Quantitative and algorithmic aspects of concurrency"

- Olivier Bodini, Matthieu Dien, Antoine Genitrini, Martin Pépin, Frédéric Peschanski, ...
- Identify **fundamental** components of concurrency and **interpret** them as combinatorial objects
- Algorithmic solutions for the **counting** and **sampling** problems
- Analytical results (when possible)

Outline

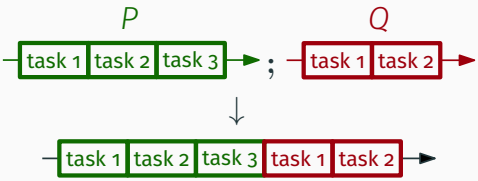
Fork-Join parallelism

Parallel composition



Execution = any interleaving of an execution of P and an execution of Q.

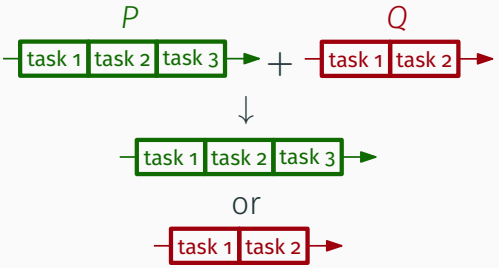
Sequential composition



Execution = an execution of P followed by an execution of Q.

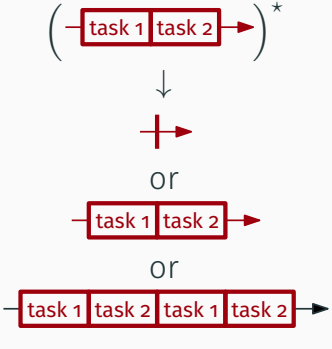
Non-determinism and loops

Non-deterministic choice



Execution = an execution of P or an execution of Q.

Loop



Execution = sequence of executions of Q

Non-deterministic Fork-Join programs (NFJ)

$$
\begin{array}{rcll}
P, Q & ::= & P \parallel Q & \text{(parallel composition)} \\
& | & P; Q & \text{(sequential composition)} \\
& | & P + Q & \text{(non-deterministic choice)} \\
& | & P^{\star} & \text{(loop)} \\
& | & a & \text{(atomic action)} \\
& | & 0 & \text{(empty program)}
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> All atomic actions accross a program are **distinct**.

Combinatorial interpretation

Define the executions of P as a combinatorial class $[\![P]\!]$:

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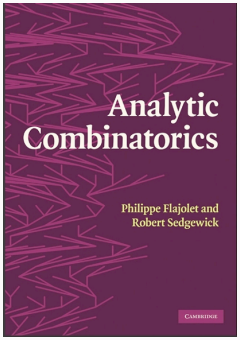

"If you can specify it, you can analyse it"

Outline

Counting executions

Algorithm: $P \xrightarrow{\text{prev. slide}} [\![P]\!] \xrightarrow{\text{symbolic method}} GF \xrightarrow{[z^n]} \text{count}$

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Theorem

The counting algorithm performs $O(|P|M(n))$ arithmetic operations on big integers.

The coefficients of the polynomial have $O(n \ln n)$ bits.

- $|P|$ is the syntactic size of P.
- $M(n)$ is the cost of the multiplication of two polynomials of degree n.

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$$\Longrightarrow O(|P|M(n)M(n \ln n)) \text{ bit-complexity.}$$

Random sampling — the recursive method

Algorithm: $P \xrightarrow{\text{prev. slides}} [\![P]\!] \xrightarrow[[FZC'93]]{\text{recursive method}}$ uniform execution

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$$\textsc{sample}((a+b)^\star \parallel (c+(d;e)+(f;g)), 3)$$

Rule:

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$$\textsc{sample}((a+b)^\star \parallel (c+(d;e)+(f;g)), 3)$$

Rule:

$$P_n = Q_0 R_n \binom{n}{0} + Q_1 R_{n-1} \binom{n}{1} + Q_2 R_{n-2} \binom{n}{2} + \cdots + Q_n R_0 \binom{n}{n}$$

Pick $k \in [\![0; n]\!]$ with probability $Q_k R_{n-k} \binom{n}{k} / P_n$

$$1 \cdot 0 \cdot \binom{3}{0} + 2 \cdot 2 \cdot \binom{3}{1} + 4 \cdot 1 \cdot \binom{3}{2} + 8 \cdot 0 \cdot \binom{3}{3} = 24 \cdot (0 + 1/2 + 1/2 + 0)$$

Random sampling — the recursive method

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$\textsc{shuffle}(\textsc{sample}((a+b)^\star, 1), \textsc{sample}((c+(d;e)+(f;g)), 2))$

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$$P^\star \to 0 + P; P^\star$$

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$$0 \cdot 1 + 2 \cdot 1 = 2 \cdot (0 + 1)$$

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Rule:

$P_n = Q_n + R_n$

Choose Q with probability Q_n/P_n

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$\text{SHUFFLE}(a, \ldots)$

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$$\textsc{shuffle}(a, \textsc{sample}((c + (d; e) + (f; g)), 2))$$

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dae

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Random sampling — complexity

Theorem

Random sampling of executions has complexity $O(n \cdot \min(h(P), \ln n))$ where h denotes the "height" of P i.e. its maximum number of nested constructors.

The $O(n \ln n)$ bound is an consequence of [FZC'93,Molinero'05].

The algorithms in numbers

Counting

$\lvert P\rvert$	n	# exec° $\leq n$	runtime
500	500	$1.058 \cdot 2^{1927}$	0.076s
500	1000	$1.081 \cdot 2^{3832}$	0.462s
500	3000	$1.341 \cdot 2^{11423}$	6.428s
1000	500	$1.473 \cdot 2^{2330}$	0.159s
1000	1000	$1.044 \cdot 2^{4712}$	0.874s
1000	3000	$1.092 \cdot 2^{14181}$	13.488s
2000	500	$1.768 \cdot 2^{2380}$	0.330s
2000	1000	$1.215 \cdot 2^{4746}$	1.870s
2000	3000	$1.699 \cdot 2^{14120}$	25.376s
5000	500	$1.607 \cdot 2^{2923}$	0.897s
5000	1000	$1.469 \cdot 2^{6016}$	5.434s
5000	3000	$1.226 \cdot 2^{18116}$	75.649s

Sampling

$\lvert P\rvert$	n	# exec° $= n$	UnifExec	IQR
500	500	$1.969 \cdot 2^{1926}$	0.218ms	5µs
500	1000	$1.004 \cdot 2^{3832}$	0.563ms	21µs
500	3000	$1.245 \cdot 2^{11423}$	3.718ms	203µs
1000	500	$1.420 \cdot 2^{2330}$	0.301ms	10µs
1000	1000	$1.005 \cdot 2^{4712}$	0.777ms	28µs
1000	3000	$1.051 \cdot 2^{14181}$	4.829ms	481µs
2000	500	$1.704 \cdot 2^{2380}$	0.308ms	14µs
2000	1000	$1.169 \cdot 2^{4746}$	1.021ms	51µs
2000	3000	$1.634 \cdot 2^{14120}$	7.291ms	1.2ms
5000	500	$1.589 \cdot 2^{2923}$	0.309ms	7µs
5000	1000	$1.448 \cdot 2^{6016}$	0.898ms	43µs
5000	3000	$1.208 \cdot 2^{18116}$	18.526ms	1.5ms

Outline

Conclusion

Take away:

- Applying methods from analytic combinatorics to concurrent problem can be effective

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Implementation

- `https://gitlab.com/ParComb/libnfj`

Future work:

- Statistical model-checking
- Generalize the model

Thank you for your attention!

Execution prefixes

Program P	Prefix program $\mathrm{pref}(P)$	Specification of the prefixes $\langle P \rangle$
0	0	$\mathcal{E}$
a	$0 + a$	$\mathcal{E} + \mathcal{Z}$
$P \parallel Q$	$(\mathrm{pref}(P) \parallel \mathrm{pref}(Q))$	$\langle P \rangle \star \langle Q \rangle$
$P; Q$	$\mathrm{pref}(P) + (P; \mathrm{pref}(Q))$	$\langle P \rangle + S(P) \times (\langle Q \rangle \setminus \mathcal{E})$
$P + Q$	$\mathrm{pref}(P) + \mathrm{pref}(Q)$	$\langle P \rangle + (\langle Q \rangle \setminus \mathcal{E})$
$P^\star$	$P^\star; \mathrm{pref}(P)$	$\mathcal{E} + S(P^\star) \times (\langle P \rangle \setminus \mathcal{E})$

Prefix covering

Table 1: Expected number of prefixes to be sampled to discover 20% of the prefixes of a random program of size 25 with either the isotropic or the uniform method.

Prefix length	1	2	3	4	5
# prefixes	11	18	30	60	128
Isotropic	2.1	4.45	11.17	35.09	$1.28 \cdot 10^{14}$
Uniform	2.1	3.18	6.57	13.26	27.69
Gain	0%	40%	70%	165%	$4.61 \cdot 10^{14}$%