Statistical Analysis of Non-Deterministic Fork-Join Processes

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Joint work with Antoine Genitrini & Frédéric Peschanski
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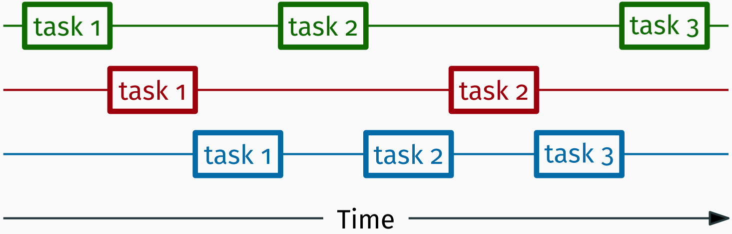
What is concurrency?



One computation unit shared by several processes:

→ Possible dependencies between processes

→ Scheduling

Why is it difficult?

You would like to check that **all possible schedulings** are correct.

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- Many possible schedulings: combinatorial explosion

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“**all schedulings**” → Combinatorics!

- Many possible schedulings: combinatorial explosion
- Can we (efficiently) count them?
- Can we (efficiently) sample among them?

Why is it difficult? (2)

Negative result (Brightwell & Winkler ’91)

Counting the linear extensions of a partial order is a *#-P* complete problem.

I.e. it is as hard as counting the number of solutions in SAT.

Why is it difficult? (2)

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Counting the linear extensions of a partial order is a *#-P* complete problem.

I.e. it is as hard as counting the number of solutions in SAT.

So we cannot count efficiently... in the **general** case.
But we can have some restrictions on the programs.

The long-term project

"Quantitative and algorithmic aspects of concurrency"

- Olivier Bodini, Matthieu Dien, Antoine Genitrini, MP, Frédéric Peschanski, ...
- Identify **fundamental** components of concurrency and **interpret** them as combinatorial objects
- Algorithmic solutions for the **counting** and **sampling** problems
- Analytical results (when possible)

Outline

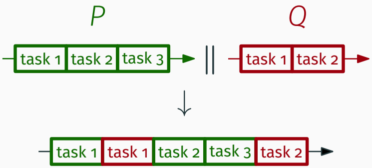
A class of concurrent programs

Algorithmic aspects

Conclusion and perspective

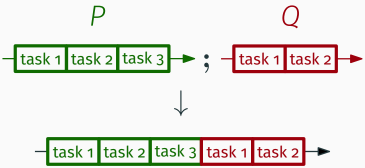
Fork-Join parallelism

Parallel composition



Execution = any interleaving of an execution of *P* and an execution of *Q*.

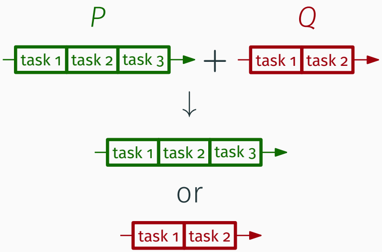
Sequential composition



Execution = an execution of *P* followed by an execution of *Q*.

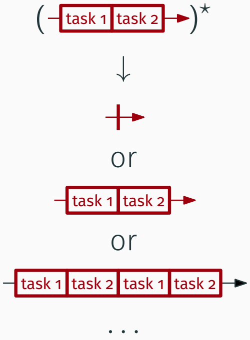
Non-determinism and loops

Non-deterministic choice



Execution = an execution of P or an execution of Q.

Loop



Execution = sequence of executions of Q

Non-deterministic Fork-Join programs (NFJ)

$$
\begin{array}{rcll}
P, Q & ::= & P \parallel Q & \text{(parallel composition)} \\
& \mid & P; Q & \text{(sequential composition)} \\
& \mid & P + Q & \text{(non-deterministic choice)} \\
& \mid & P^\star & \text{(loop)} \\
& \mid & a & \text{(atomic action)} \\
& \mid & 0 & \text{(empty program)}
\end{array}
$$

Combinatorial interpretation

Define the executions of P as a combinatorial class $[\![P]\!]$:

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- Labelled and unlabelled operators in the same grammar;

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$$\cancel{[\![P^\star]\!] = \textsc{Seq}\left([\![P]\!]\right)}$$ ⚠

$$[\![P+Q]\!]_{\neq 0} = [\![P]\!]_{\neq 0} + [\![Q]\!]_{\neq 0}$$ ●

$$[\![P^\star]\!] = \textsc{Seq}\left([\![P]\!]_{\neq 0}\right)$$ ●

- ● Labelled and unlabelled operators in the same grammar;
- ⚠ $[\![P]\!]$, $[\![Q]\!]$ might contain the empty execution;
- ● $[\![P]\!]_{\neq 0} =$ non-empty executions of P.

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Algorithm: $P \xrightarrow{\text{prev. slide}} [\![P]\!] \xrightarrow{\text{symbolic method}} GF \xrightarrow{[z^n]} \text{count}$

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\textsc{count}(P; Q) &= p(z) \cdot q(z) \\
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All operation are taken $\bmod z^{n+1}$

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- $p(z) \circledcirc q(z) = \mathcal{L}\left(\mathcal{B}\left(p(z)\right) \cdot \mathcal{B}\left(q(z)\right)\right)$
  where $\mathcal{L}\left(\sum_n \frac{a_n}{n!} z^n\right) = \sum_n a_n z^n$ and $\mathcal{B}\left(\sum_n a_n z^n\right) = \sum_n \frac{a_n}{n!} z^n$

Counting executions — complexity

Theorem

The counting algorithm performs $O(|P|M(n))$ arithmetic operations on big integers.

The coefficients of the polynomial have $O(n \ln n)$ bits.

- $|P|$ is the syntactic size of P.
- $M(n)$ is the cost of the multiplication of two polynomials of degree n.

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$\implies$ $O(|P|M(n)M(n \ln n))$ bit-complexity.

Random sampling of executions

Algorithm: $P \xrightarrow{\text{prev. slides}} [\![P]\!] \xrightarrow[\text{[}FZC'93\text{]}]{\text{recursive method}}$ uniform execution

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$$\textsc{sample}((a+b)^\star \parallel (c + (d;e) + (f;g)), 3)$$

Rule:

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$$\textsc{sample}\big((a+b)^\star \parallel (c + (d;e) + (f;g)), 3\big)$$

Rule: $P_n = Q_0 R_0 \binom{n}{0} + Q_1 R_{n-1} \binom{n}{1} + Q_2 R_{n-2} \binom{n}{2} + \cdots$
Pick $k \in [\![0;n]\!]$ with probability $Q_k R_{n-k} \binom{n}{k} / P_n$

$$1 \cdot 0 \cdot \binom{3}{0} + 2 \cdot 2 \cdot \binom{3}{1} + 4 \cdot 1 \cdot \binom{3}{2} + 8 \cdot 0 \cdot \binom{3}{3} = 24 \cdot (0 + 1/2 + 1/2 + 0)$$

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$$\text{SHUFFLE}(\text{SAMPLE}((a+b)^\star, 1), \text{SAMPLE}((c+(d;e)+(f;g)), 2))$$

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Pick $k \in [\![0;n]\!]$ with probability Q_kR_{n-k}/P_n

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Rule: $P_n = Q_n + R_n$
Choose Q with probability Q_n/P_n

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$$\frac{}{dae}$$

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$P_n = Q_0 R_n + Q_1 R_{n-1} + Q_2 R_{n-2} + \cdots + Q_n R_0$

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Solution 2 (boustrophedonic alg., [FZC'93,Molinero'05]):

Draw $x \sim \textsc{Unif}([\![0; P_n[\![)$ and take the minimum k such that $x < Q_0 R_n + Q_n R_0 + Q_1 R_{n-1} + Q_{n-1} R_1 + Q_2 R_{n-2} + \ldots$ (k terms).

Random sampling of executions — complexity

Theorem [FZC'93,Molinero'05]

The recursive method with solution 2 has complexity $O(n \ln n)$.

Theorem [Molinero'05]

The recursive method has complexity $O(n)$ on recursion-free specifications.

Random sampling of executions — complexity

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The constant hidden in the O depends on the specification!

Random sampling of executions — complexity

Theorem [FZC'93,Molinero'05]

The recursive method with solution 2 has complexity $O(n \ln n)$.

Theorem (updated)

The recursive method has complexity $O(h \cdot n)$ on recursion-free specification, where h is the number of nested operators of the spec.

The constant hidden in the O depends on the specification!

Random sampling of executions — complexity

Theorem

Random sampling of executions has complexity $O(n \cdot \min(h(P), \ln n))$ where h denotes the "height" of P i.e. its maximum number of nested constructors.

Outline

Conclusion

Take away:

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Future work:

- Generalize the model
- Analytic properties of the OGF of $[\![P]\!]$?
- Statistical model-checking

Thanks for your attention