DIRECTED ORDERED ACYCLIC GRAPHS

ASYMPTOTIC ANALYSIS AND EFFICIENT RANDOM SAMPLING

Martin Pépin
joint work with Antoine Genitrini & Alfredo Viola

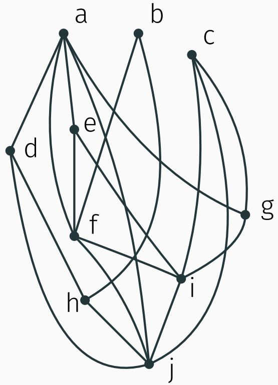
March 21, 2023
Séminaire algo — Caen

LIPN

Directed Acyclic Graphs

Directed Acyclic Graph (DAG)

- A finite set of vertices V e.g. $\{a, b, c, \ldots, j\}$;
- a set of directed edges $E \subseteq V \times V$;
- no cycles.



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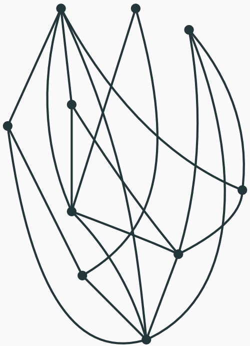

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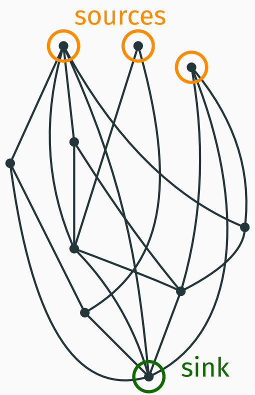

Why DAGs?

Omnipresent data structure:

- Encoding partial orders in scheduling problems;
- Git histories;
- genealogy trees (those are not trees!);
- compacted trees (or XML documents, etc.);
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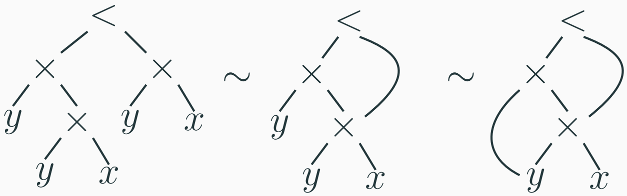

State of the art

Labelled DAGs:

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- ● Inclusion-exclusion
- ● No or little control over the number of edges
- ● Only binary

Outline of the presentation

Background

Directed ordered acyclic graphs

↳ *definition and recursive decomposition*

Asymptotic analysis

↳ *matrix encoding*
↳ *asymptotic result*
↳ *faster sampler*

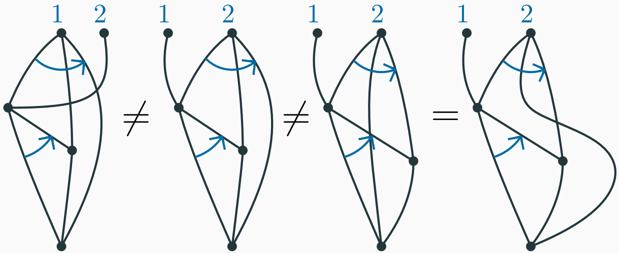
Labelled DAGs

↳ *a new way of counting*

A new kind of DAG

Directed Ordered Acyclic Graphs

DOAG = Unlabelled DAG
+ a total order on the **outgoing** edges of each vertex
+ a total order on the sources



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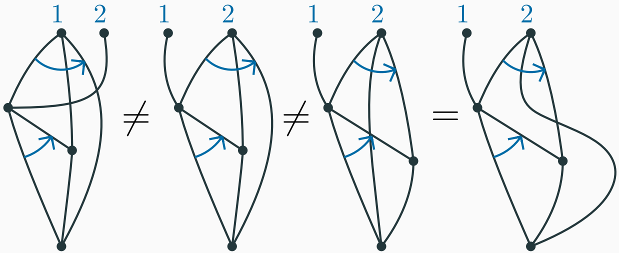
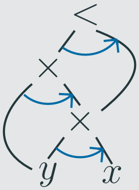
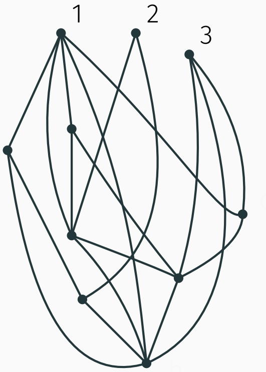


Motivation



Recursive decomposition



n vertices, m edges, k sources

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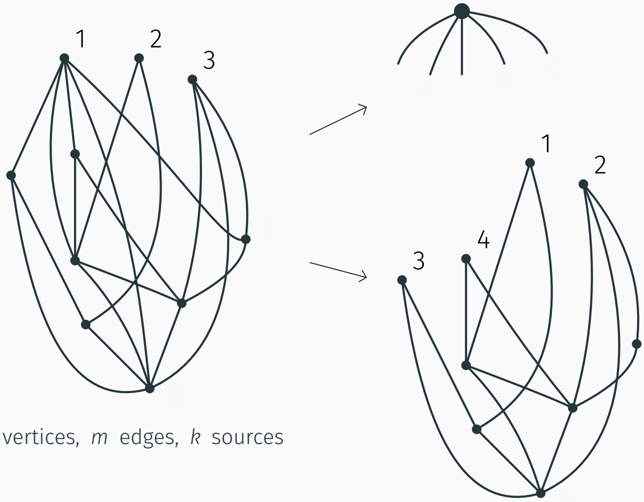

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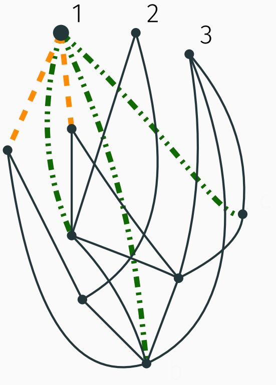

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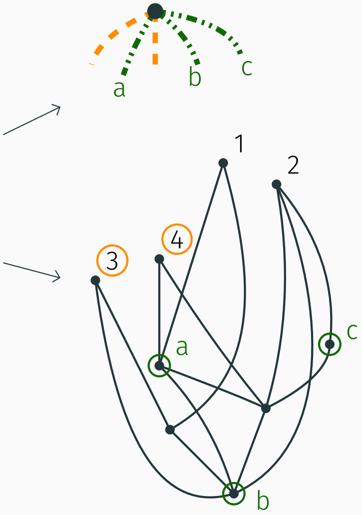

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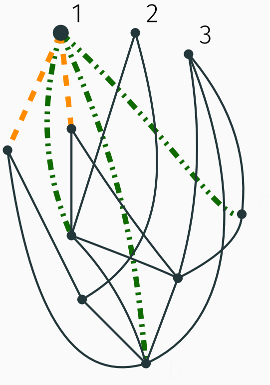

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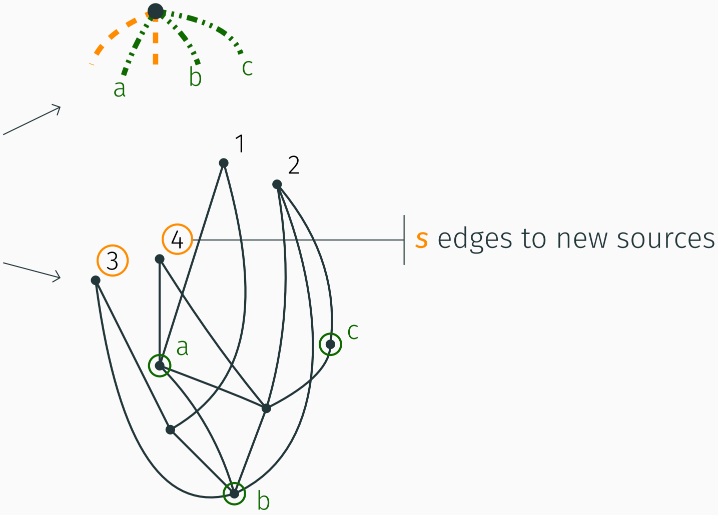

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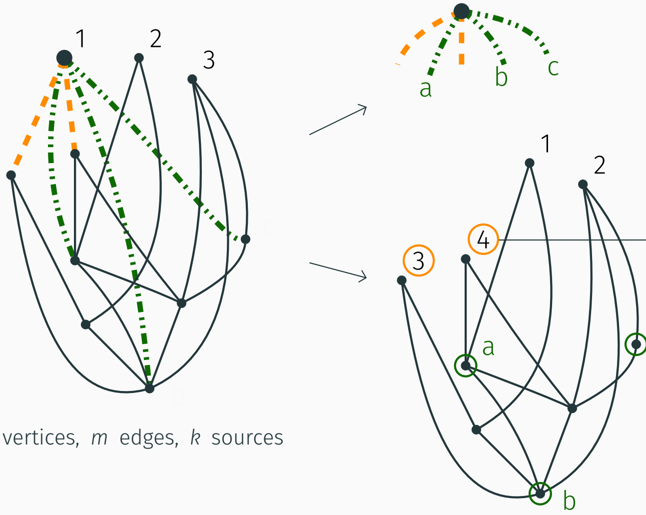

n vertices, m edges, k sources

s edges to new sources

i edges to internal nodes

↳ $\binom{n-k-s}{i}$ choices

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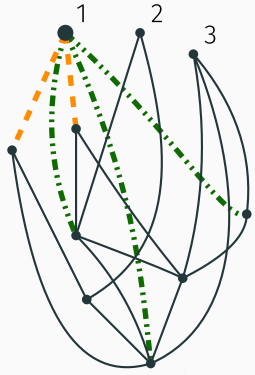

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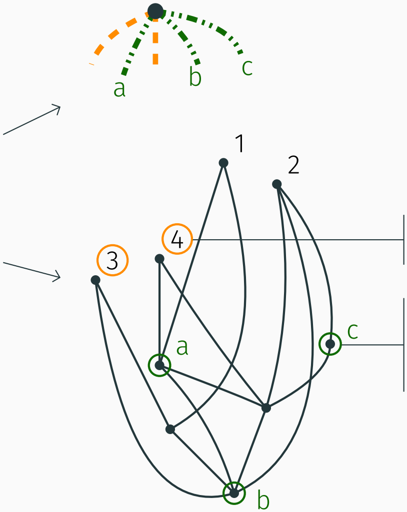

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$(n-1)$ vertices, $(m-i-s)$ edges, $(k+s-1)$ sources

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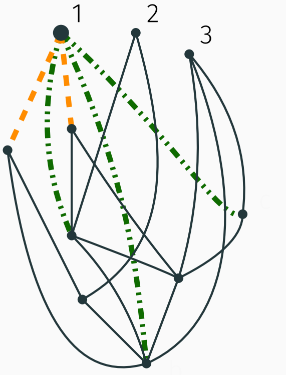

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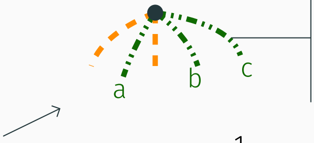
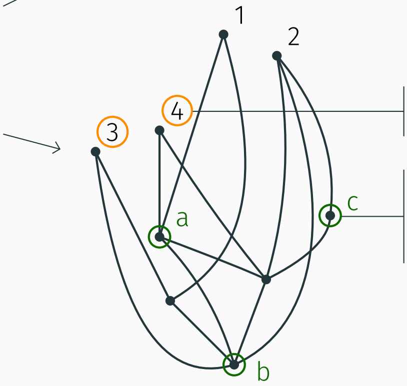


permute the edges
interleave with
↳ $i!\binom{i+s}{i}$ choices



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Recurrence formula

Counting formula

$$\begin{aligned} D_{n,m,k} &= \#\{\text{DOAGs with } n \text{ vertices, } m \text{ edges and } k \text{ sources}\} \\ &= \sum_{i,s\geq 0} D_{n-1,m-i-s,k+s-1} \binom{n-k-s}{i} i! \binom{i+s}{i} \end{aligned}$$

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→ $O(N^4 M)$ operations;
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In practice: for $M \approx 400$, one sink → counting = several minutes
→ sampling = a few ms

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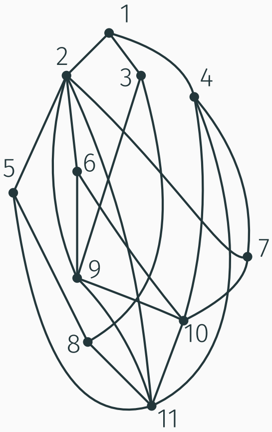
Number of single-source DOAGs (P., Viola, 2023+)

$$D_n \underset{n\to\infty}{\sim} c \cdot n^{-1/2} \cdot e^{n-1} \cdot ¡n - 1!$$

for $c \approx 0.4967$ *and where* $¡x! = \prod_{k=1}^{x} k!$.

Matrix encoding



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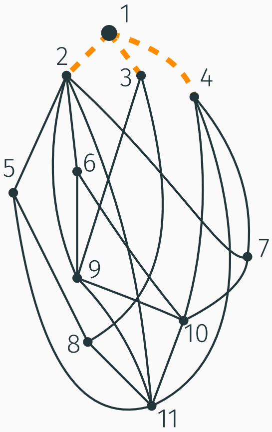

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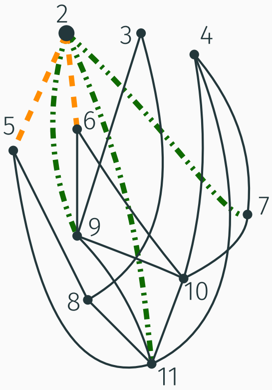

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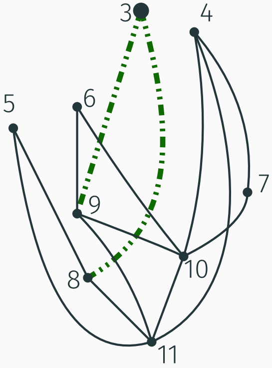

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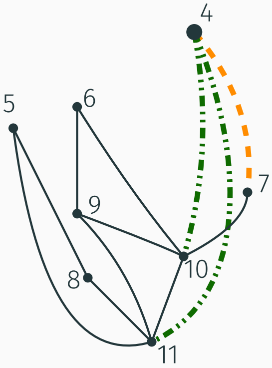

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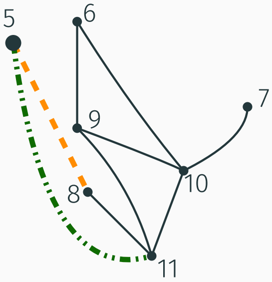

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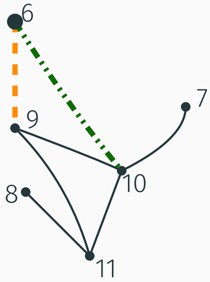

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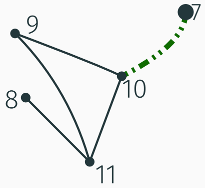

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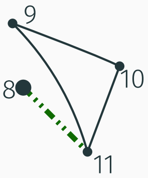

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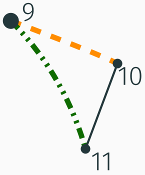

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Matrix encoding

1. strict upper triangular matrix;

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3. $a_{1,2} \neq 0$;

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11											

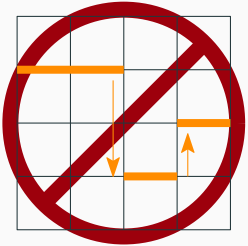
Matrix encoding

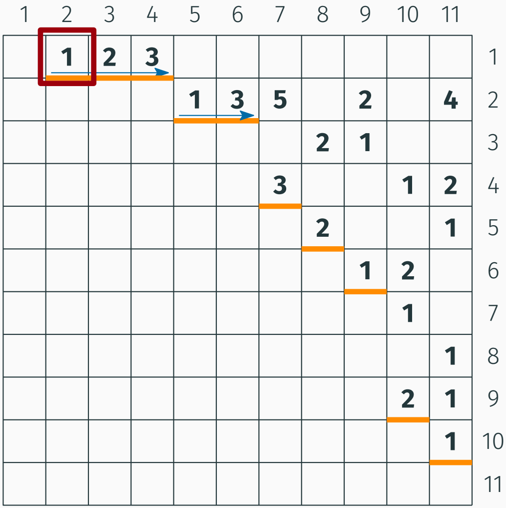
1. strict upper triangular matrix;
2. lines use an interval of values;
3. $a_{1,2} \neq 0$;
4. increasing numbers above orange lines;

1	2	3	4	5	6	7	8	9	10	11	
	1	**2**	**3**								1
				1	**3**	**5**		**2**		**4**	2
							2	**1**			3
						3			**1**	**2**	4
							2			**1**	5
								1	**2**		6
									1		7
										1	8
									2	**1**	9
										1	10
											11

Matrix encoding

1. strict upper triangular matrix;
2. lines use an interval of values;
3. $a_{1,2} \neq 0$;
4. increasing numbers above orange lines;
5. orange lines go down.





Proof sketch (1/3)

The plan: 1. Upper bound 2. Lower bound 3. Bootstrapping

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

$$\boxed{6}\,\boxed{1}\,\boxed{}\,\boxed{5}\,\boxed{}\,\boxed{2}\,\boxed{4}\,\boxed{}\,\boxed{3} = \boxed{6}\,\boxed{1}\,\boxed{5}\,\boxed{2}\,\boxed{4}\,\boxed{3} \shuffle \boxed{}\,\boxed{}\,\boxed{}$$

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

6	**1**		**5**		**2**	**4**		**3**	$=$	**6**	**1**	**5**	**2**	**4**	**3**	$⧢$			

$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

$$\boxed{6}\boxed{1}\boxed{}\boxed{5}\boxed{}\boxed{2}\boxed{4}\boxed{}\boxed{3} = \boxed{6}\boxed{1}\boxed{5}\boxed{2}\boxed{4}\boxed{3} ш \boxed{}\boxed{}\boxed{}$$

$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

$$V(z) = (1-z)^{-1} e^{z}$$

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

$$\boxed{6}\boxed{1}\boxed{}\boxed{5}\boxed{}\boxed{2}\boxed{4}\boxed{}\boxed{3} = \boxed{6}\boxed{1}\boxed{5}\boxed{2}\boxed{4}\boxed{3} ⧢ \boxed{}\boxed{}\boxed{}$$

$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

$$V(z) = (1-z)^{-1}e^{z}$$

$$v_n = e \cdot n! - o(1)$$

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

6	1		5		2	4		3

$=$

6	1	5	2	4	3

$\sqcup\!\sqcup$

$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

$$V(z) = (1-z)^{-1}e^{z}$$

$$v_n = e \cdot n! - o(1)$$

$\#\{\text{DOAG matrices}\} = \#\{\text{collections of rows}\} \leq \#\{\text{collections of variations}\}$

Proof sketch (1/3)

The plan: **1. Upper bound** 2. Lower bound 3. Bootstrapping

6	1		5		2	4		3	=	6	1	5	2	4	3	⧢			

$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

$$V(z) = (1-z)^{-1} e^{z}$$

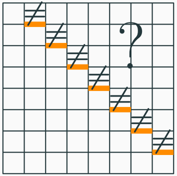
$$v_n = e \cdot n! - o(1)$$

$\#\{\text{DOAG matrices}\} = \#\{\text{collections of rows}\} \leq \#\{\text{collections of variations}\}$

$$D_n \leq \prod_{k=1}^{n-1} v_k \leq e^{n-1} \text{ ¡} n - 1 \text{!}$$

Proof sketch (2/3)

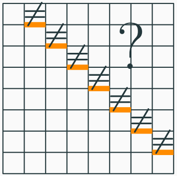
The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$ 

(constraints are automatically satisfied)

Proof sketch (2/3)

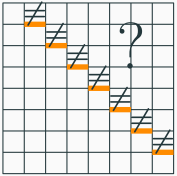
The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

{DOAG matrices} ⊇  (constraints are automatically satisfied)

$\# \boxed{\neq \,|\, ? \,|\, ? \,|\, ? \,|\, ? \,|\, ? \,|\, ? \,|\, ? \,|\, ?} = V_k - V_{k-1}$

Proof sketch (2/3)

The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

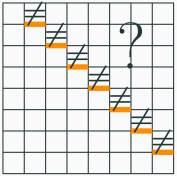
{DOAG matrices} $\supseteq$ 

(constraints are automatically satisfied)

$$\#\,\boxed{\neq}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?} = v_k - v_{k-1} = e \cdot k! \cdot \left(1 - \frac{1}{k} - o\left(\frac{1}{(k-1)!}\right)\right)$$

Proof sketch (2/3)

The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$ 

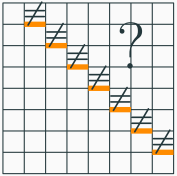
(constraints are automatically satisfied)

$$\#\boxed{\neq}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?} = v_k - v_{k-1} = e \cdot k! \cdot \left(1 - \frac{1}{k} - o\left(\frac{1}{(k-1)!}\right)\right)$$

$$D_n \geq e^{n-1}\ ¡n - 1! \prod_{k=2}^{n-1} \left(\frac{k-1}{k} + o\left(\frac{1}{(k-1)!}\right)\right)$$

Proof sketch (2/3)

The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

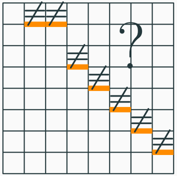
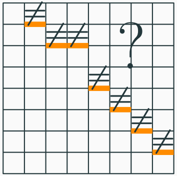
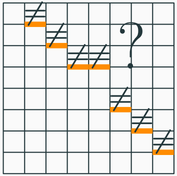
{DOAG matrices} $\supseteq$  (constraints are automatically satisfied)

$$\# \boxed{\neq}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?} = v_k - v_{k-1} = e \cdot k! \cdot \left(1 - \frac{1}{k} - o\left(\frac{1}{(k-1)!}\right)\right)$$

$$D_n \geq e^{n-1} \text{ ¡}n - 1! \prod_{k=2}^{n-1} \left(\frac{k-1}{k} + o\left(\frac{1}{(k-1)!}\right)\right) \geq e^{n-1} \text{ ¡}n - 1! \frac{A}{n} \quad \text{for some } A > 0$$

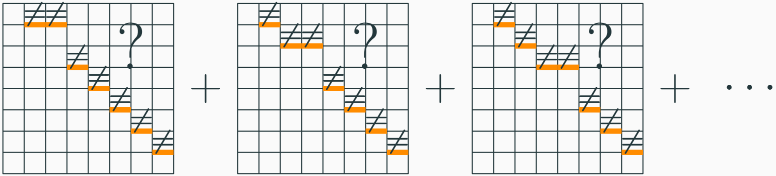
Proof sketch (2'/3)

The plan: 1. Upper bound **2'. Better lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$  $+$  $+$  $+ \cdots$

The plan: 1. Upper bound **2'. Better lower bound** 3. Bootstrapping

$\{\text{DOAG matrices}\} \supseteq$



$$D_n \geq \frac{A' \cdot \ln(n)}{n} \, e^{n-1} \, ¡n - 1!$$

The plan: 1. Upper bound **2'. Better lower bound** 3. Bootstrapping

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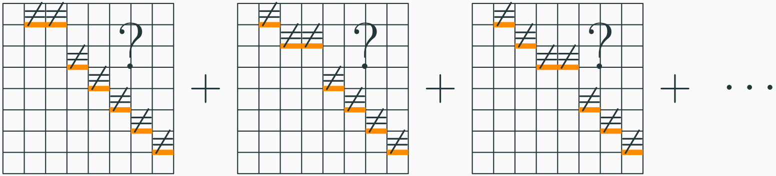
{DOAG matrices} ⊇

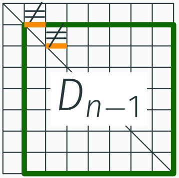


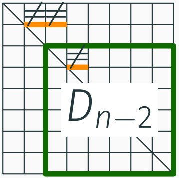
$$D_n \geq \frac{A' \cdot \ln(n)}{n} e^{n-1} ¡n-1!$$

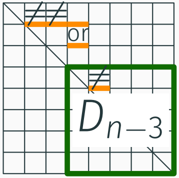
$$P_n = \frac{D_n}{e^{n-1} ¡n-1!} \quad \Rightarrow \quad \frac{A' \cdot \ln(n)}{n} \leq P_n \leq 1$$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

{DOAG matrices} = 

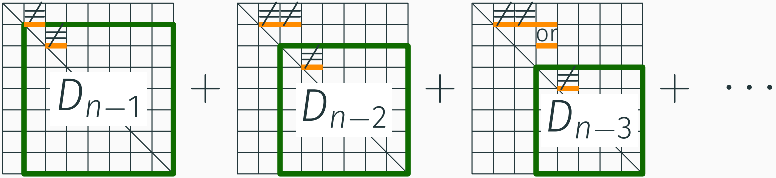
+ 

+ 

+ $\cdots$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

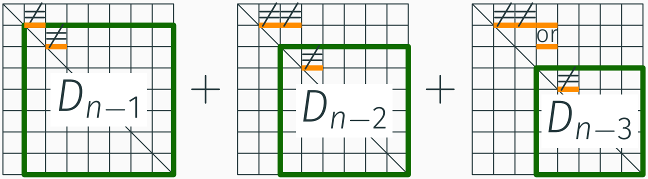
{DOAG matrices} = 

$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

{DOAG matrices} =



+ ⋯

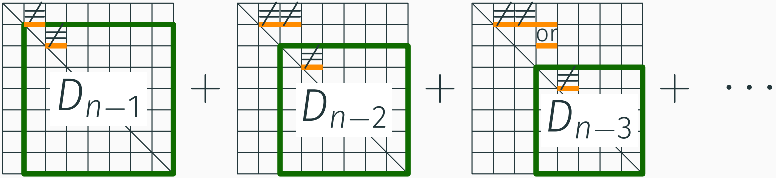
$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

$$P_n = \left(1 - \frac{1}{n-1}\right)P_{n-1} + \frac{1}{2(n-2)}\left(1 - \frac{2}{n-1} + \frac{1}{(n-1)(n-2)}\right)P_{n-2} + \cdots$$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

{DOAG matrices} =



$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

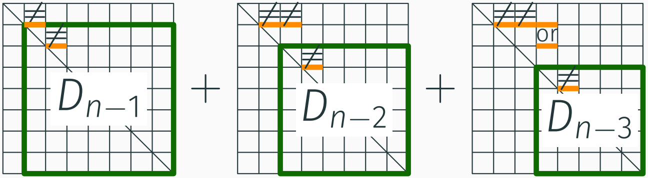
$$P_n = \left(1 - \frac{1}{n-1}\right) P_{n-1} + \frac{1}{2(n-2)} \left(1 - \frac{2}{n-1} + \frac{1}{(n-1)(n-2)}\right) P_{n-2} + \cdots$$

$$P_n \sim P_{n-1}$$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

{DOAG matrices} =



+ ⋯

$$D_n = (v_{n-1} - v_{n-2}) D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3}) v_{n-3} D_{n-2} + \cdots$$

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Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

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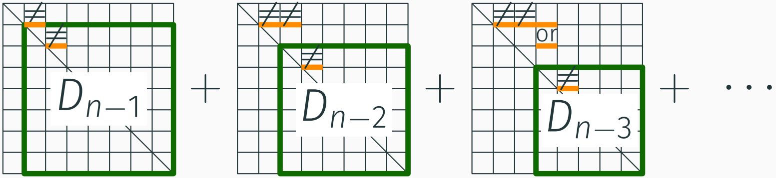


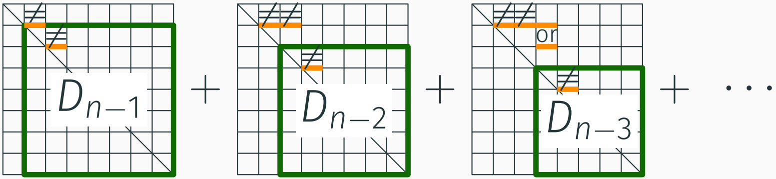
$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

$$P_n = \left(1 - \frac{1}{n-1}\right) P_{n-1} + \frac{1}{2(n-2)} \left(1 - \frac{2}{n-1} + \frac{1}{(n-1)(n-2)}\right) P_{n-2} + \cdots$$

$$P_n = P_{n-1}\left(1 - \frac{1}{2n} + O\left(n^{-2}\right)\right)$$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

{DOAG matrices} = 

$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

$$P_n = \left(1 - \frac{1}{n-1}\right) P_{n-1} + \frac{1}{2(n-2)} \left(1 - \frac{2}{n-1} + \frac{1}{(n-1)(n-2)}\right) P_{n-2} + \cdots$$

$$P_n = P_{n-1}\left(1 - \frac{1}{2n} + O\left(n^{-2}\right)\right) \Rightarrow \boxed{P_n \sim c \cdot n^{-1/2}}$$

Random sampling again!

Corollary

$$\frac{D_n}{\#\{\textit{matrices of variations of sizes } 1, 2, \ldots, n-1\}} \sim c \cdot n^{-\frac{1}{2}}$$

Random sampling again!

Corollary

$$\frac{D_n}{\#\{\text{matrices of variations of sizes } 1, 2, \ldots, n-1\}} \sim c \cdot n^{-\frac{1}{2}}$$

Rejection sampling: draw variation matrices until they correspond to a DOAG

Random sampling again!

Corollary

$$\frac{D_n}{\#\{\textit{matrices of variations of sizes } 1, 2, \ldots, n-1\}} \sim c \cdot n^{-\frac{1}{2}}$$

Rejection sampling: draw variation matrices until they correspond to a DOAG

(Naive) complexity: $\#\text{rejections} \times \text{Cost}(\text{one generation})$

Random sampling again!

Corollary

$$\frac{D_n}{\#\{\textit{matrices of variations of sizes } 1, 2, \ldots, n-1\}} \sim c \cdot n^{-\frac{1}{2}}$$

Rejection sampling: draw variation matrices until they correspond to a DOAG

(Naive) complexity: $\#$rejections $\times$ Cost(one generation)

Generating one variation: $\sim n \log_2(n)$ random bits.

Random sampling again!

Corollary

$$\frac{D_n}{\#\{\textit{matrices of variations of sizes } 1, 2, \ldots, n-1\}} \sim c \cdot n^{-\frac{1}{2}}$$

Rejection sampling: draw variation matrices until they correspond to a DOAG

(Naive) complexity: $O(\sqrt{n} \cdot n^2 \ln(n))$ random bits

Generating one variation: $\sim n \log_2(n)$ random bits.

Random sampling again!

Corollary

$$\frac{D_n}{\#\{\text{matrices of variations of sizes } 1, 2, \ldots, n-1\}} \sim c \cdot n^{-\frac{1}{2}}$$

Rejection sampling: draw variation matrices until they correspond to a DOAG

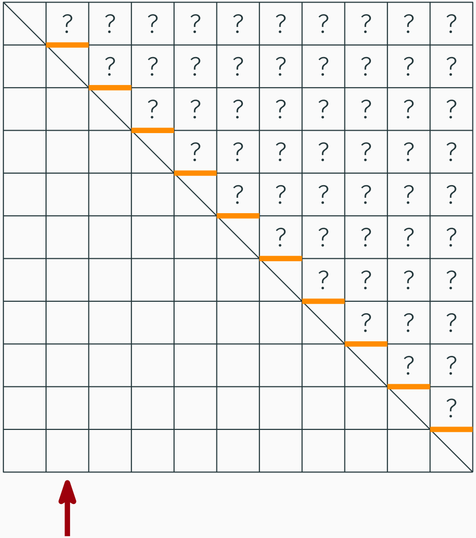
(Naive) complexity: $O(\sqrt{n} \cdot n^2 \ln(n))$ random bits

Generating one variation: $\sim n \log_2(n)$ random bits.

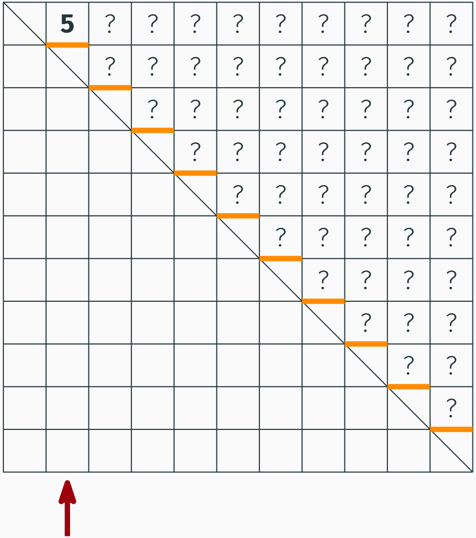
Better complexity:

$$\begin{aligned} &\text{Cost}(\text{one full generation}) + \#\text{rejections} \times \text{Cost}(\text{one failed generation}) \\ &= \frac{n^2}{2} \log_2(n) + O\left(\sqrt{n} \cdot \textbf{Cost(}\text{one failed generation}\textbf{)}\right) \end{aligned}$$

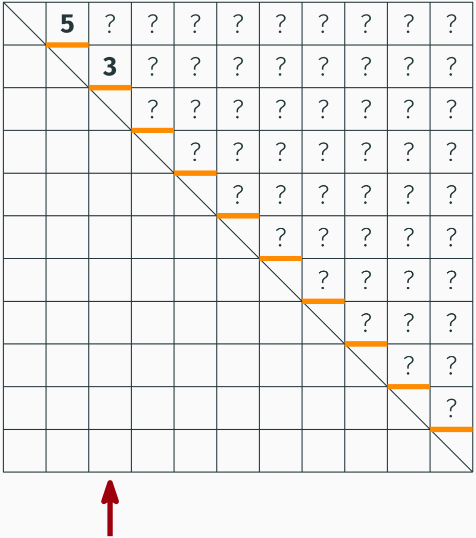
Anticipated rejection



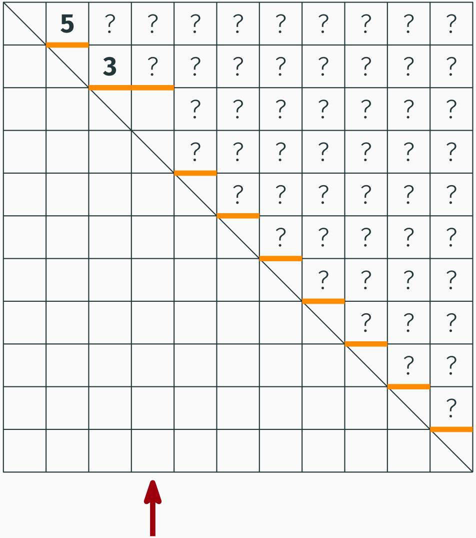
Anticipated rejection



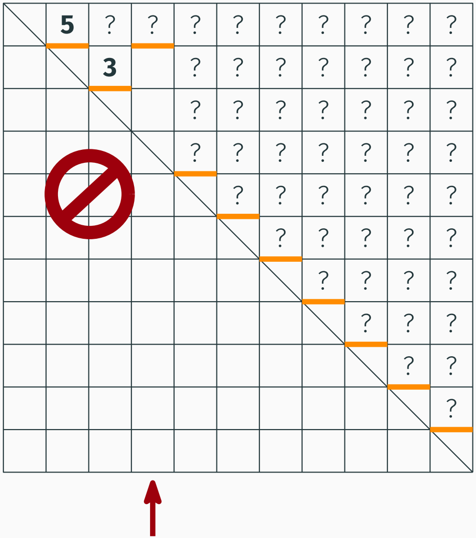
Anticipated rejection



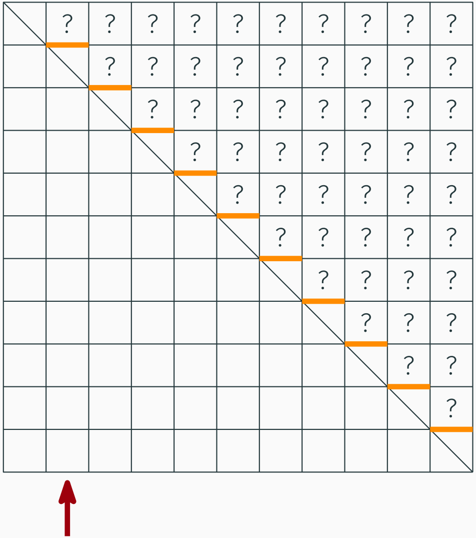
Anticipated rejection



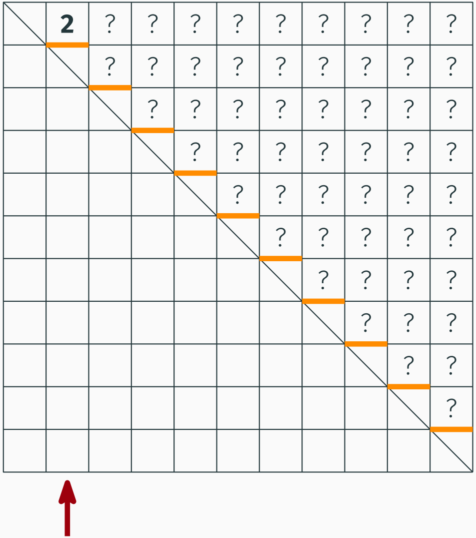
Anticipated rejection



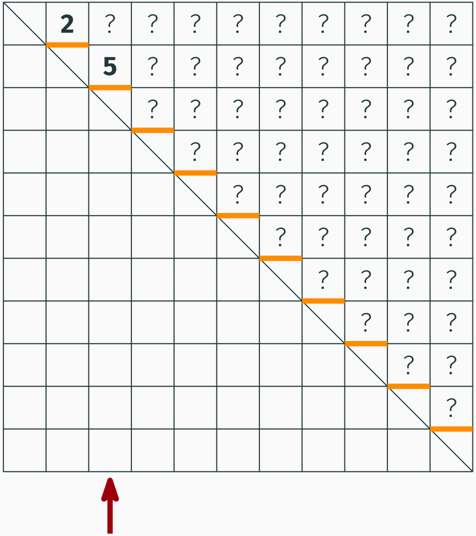
Anticipated rejection



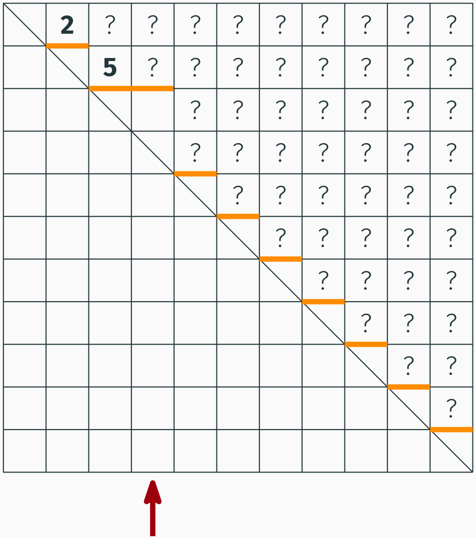
Anticipated rejection



Anticipated rejection



Anticipated rejection



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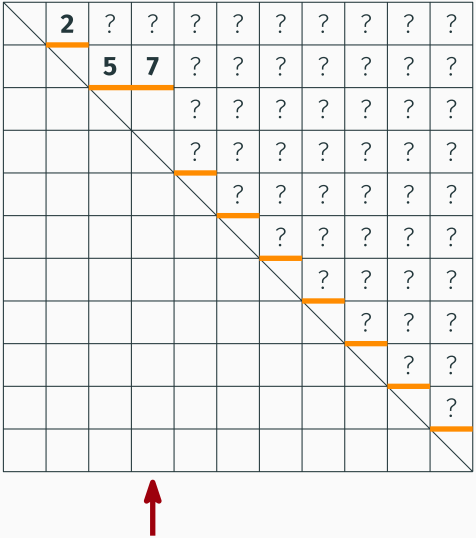
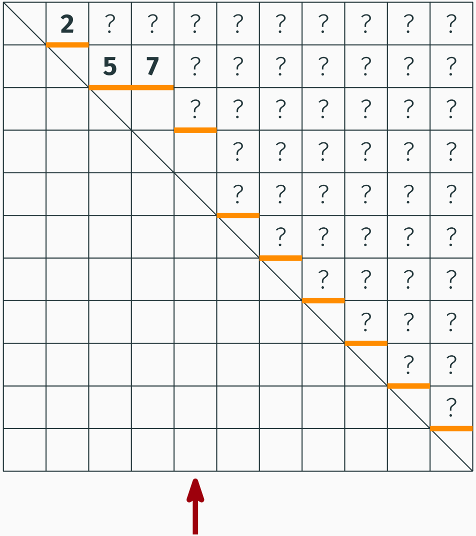
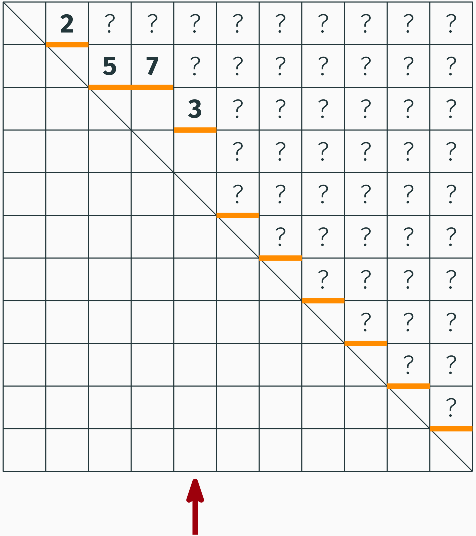


Anticipated rejection



Anticipated rejection



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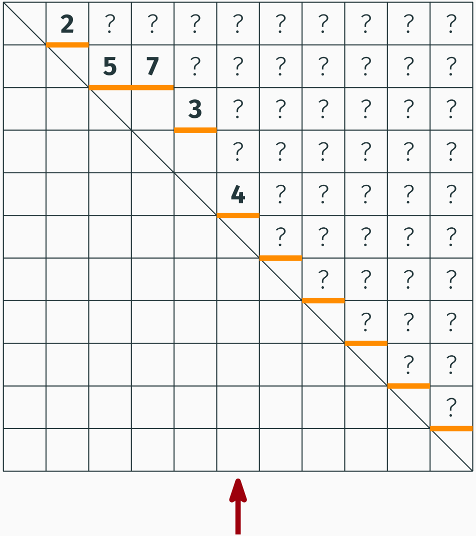
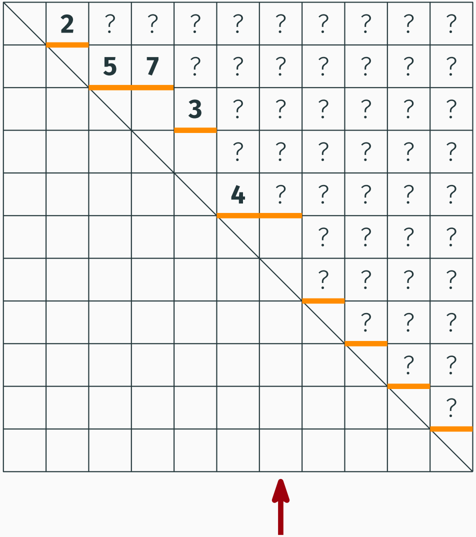
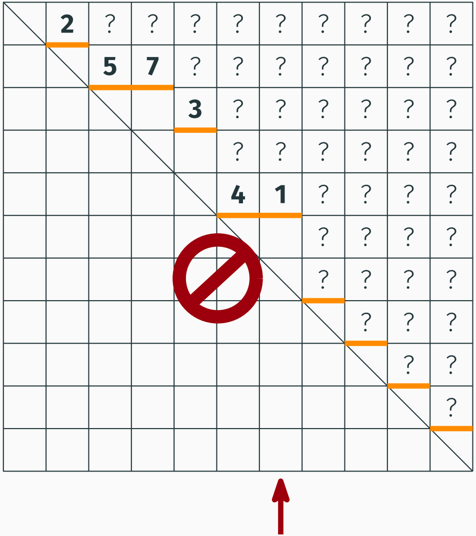


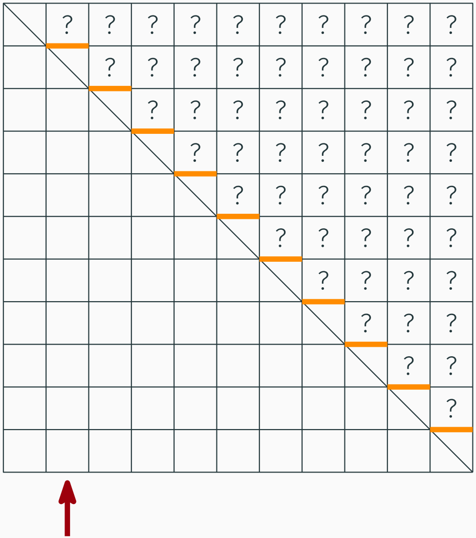
Anticipated rejection



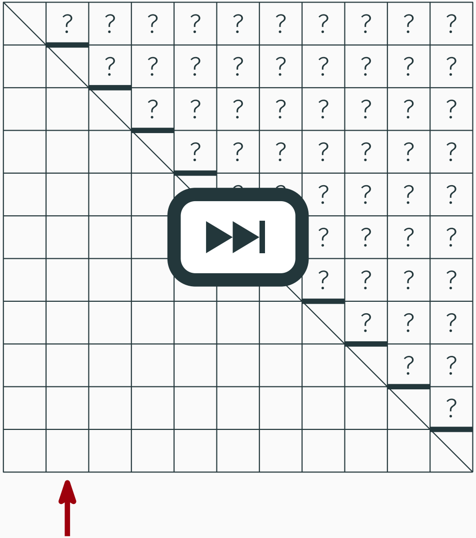
Anticipated rejection



Anticipated rejection



Anticipated rejection



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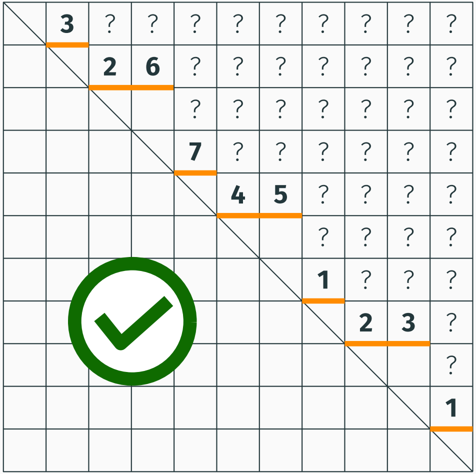

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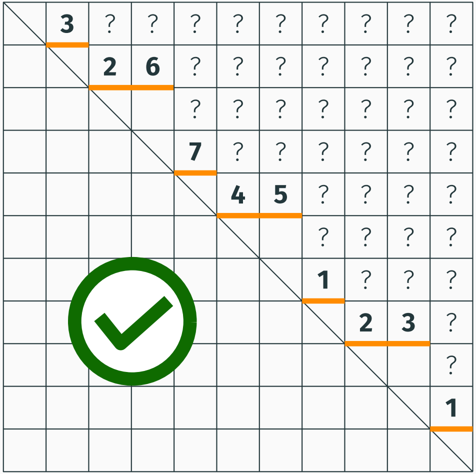

Complexity $= O(n \ln(n))$

Total complexity $= \frac{n^2}{2} \log_2(n) + O\left(\sqrt{n} \cdot n \ln(n)\right)$

Outline of the presentation

Background

Directed ordered acyclic graphs

↳ *definition and recursive decomposition*

Asymptotic analysis

↳ *matrix encoding*
↳ *asymptotic result*
↳ *faster sampler*

Labelled DAGs

↳ *a new way of counting*

But... what about labelled DAGs?

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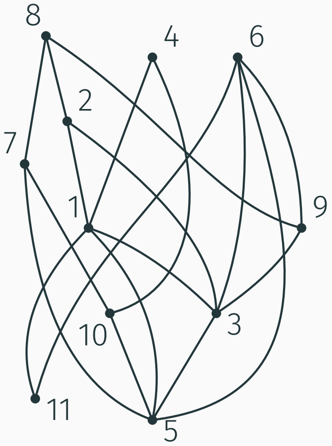
Idea: mark one source, and remove it.



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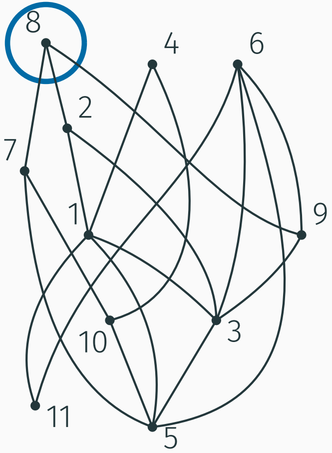

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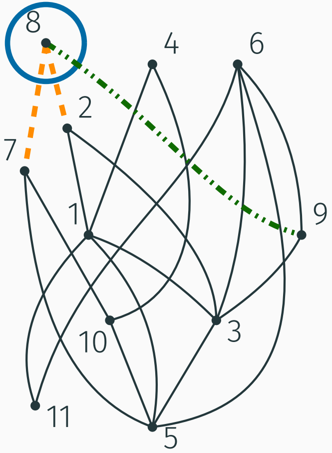

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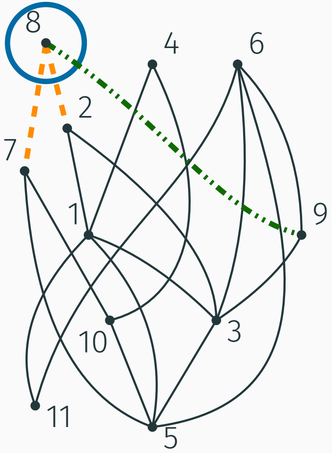

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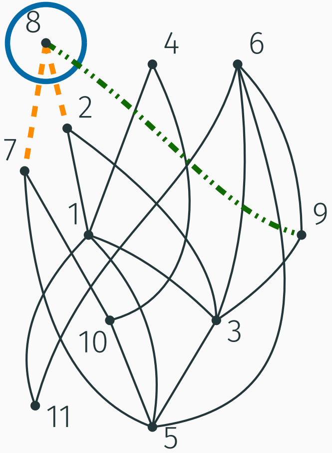

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→ New counting formula for DAGs;
→ Effective sampler with fixed number of edges and vertices.

And the matrix representation?

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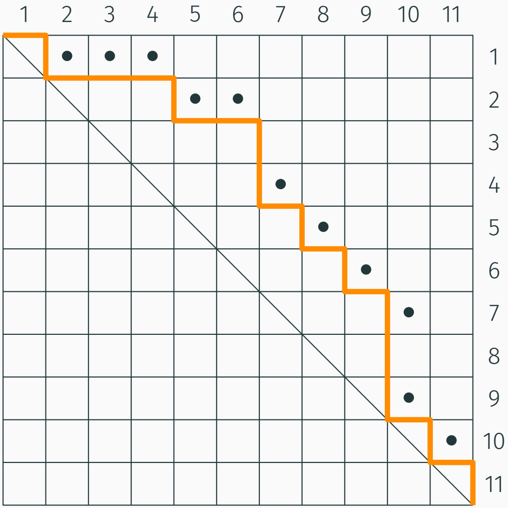
TODO

(work in progress)

And the matrix representation?

1	2	3	4	5	6	7	8	9	10	11	
	1	**2**	**3**								1
				1	**3**	**5**		**2**		**4**	2
							2	**1**			3
						3			**1**	**2**	4
							2			**1**	5
								1	**2**		6
									1		7
										1	8
									2	**1**	9
										1	10
											11

And the matrix representation?



Perspectives

- Efficient random generation of labelled DAGs
  - Collaboration with Philippe Marchal

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- What about sparse DOAGs?

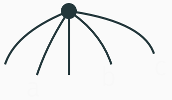
- New model: DOAG
- New way of counting
- Control over the number of edges
- When forgetting edges:
  - (Fun?) asymptotic results
  - Optimal uniform sampler
- Results on DAGs

https://github.com/Kerl13/randdag
https://wkerl.me
martin.pepin@lipn.univ-paris13.fr

Random sampling = counting

Do the same, but backwards!

Random sampling = ƃuıʇunoɔ



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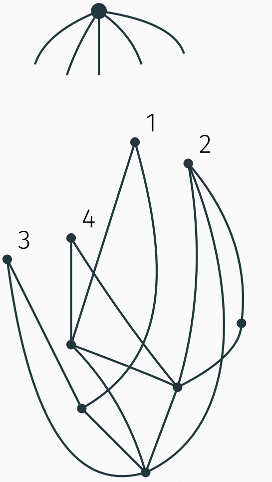
1. Select (i, s) with probability $\frac{D_{n-1,m-i-s,k+s-1}\binom{n-k-s}{i} i! \binom{i+s}{i}}{D_{n,m,k}}$;

Random sampling = ɡnıʇnuoɔ



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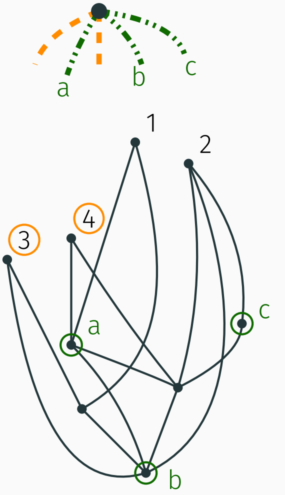

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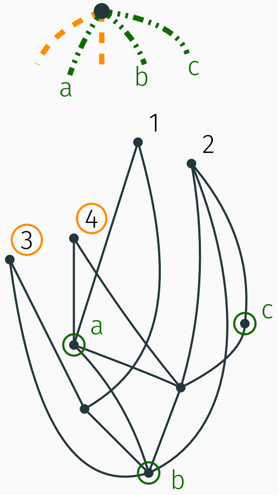
3. connect the s largest sources;

Random sampling = counting



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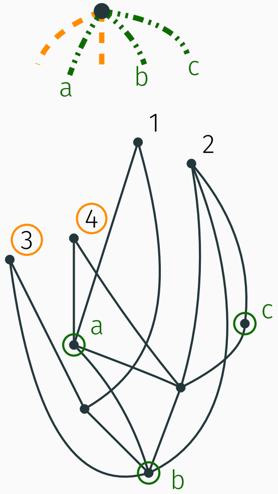

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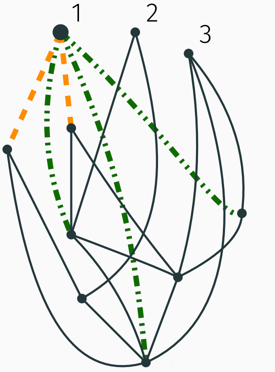
Random sampling = counting



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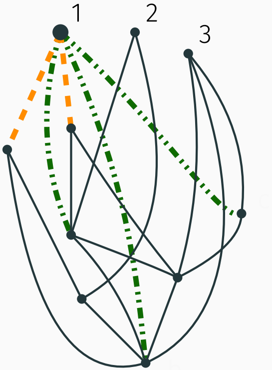

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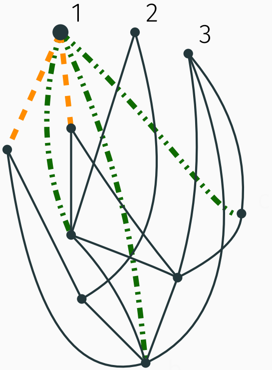
Complexity: $O\left(\sum_{v \text{ vertex}} d_v^2\right) = O(\min(N^3, M^2))$.

↳ out-degree of v

Random sampling = ɡnıʇnuoɔ



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In practice: about 400 edges in a few ms.

*1

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