DIRECTED ORDERED ACYCLIC GRAPHS

ASYMPTOTIC ANALYSIS AND EFFICIENT RANDOM SAMPLING

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joint work with Antoine Genitrini & Alfredo Viola

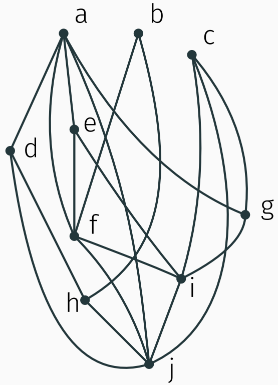
March 14, 2023
Journées ALEA 2023

LIPN

Directed Acyclic Graphs

Directed Acyclic Graph (DAG)

- A finite set of vertices V e.g. $\{a, b, c, \ldots, j\}$;
- a set of directed edges $E \subseteq V \times V$;
- no cycles.



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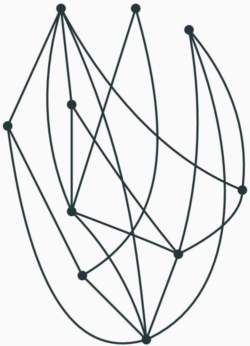

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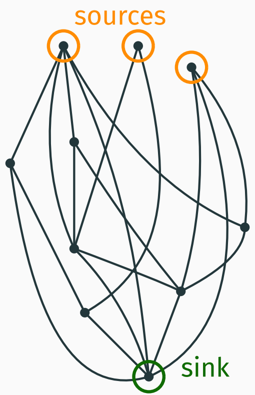

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Why DAGs?

Omnipresent data structure:

- Encoding partial orders in scheduling problems;
- Git histories;
- genealogy trees (those are not trees!);
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State of the art

Labelled DAGs:

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- ● No or little control over the number of edges
- ● Only binary

Outline of the presentation

Background

Directed ordered acyclic graphs

↳ *definition and recursive decomposition*

Asymptotic analysis

↳ *matrix encoding*
↳ *asymptotic result*
↳ *faster sampler*

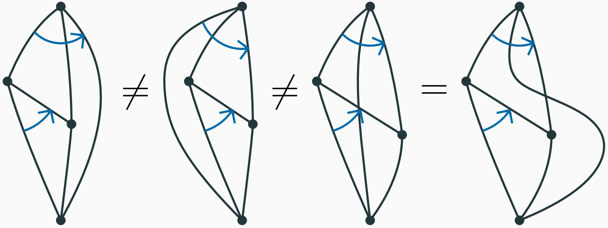
Bonus: labelled DAGs

↳ *another way of counting*

A new kind of DAG

Directed Ordered Acyclic Graphs

DOAG = Unlabelled DAG
+ a total order on the **outgoing** edges of each vertex
+ a total order on the sources



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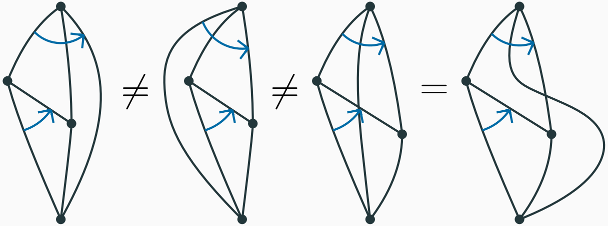
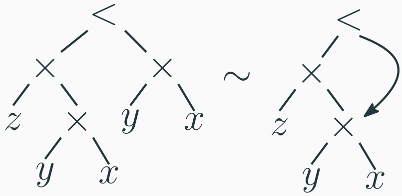
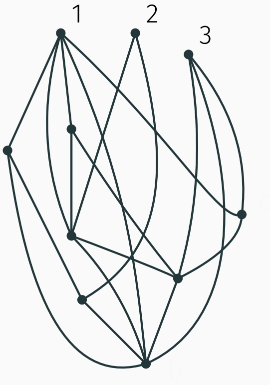




Recursive decomposition



n vertices, m edges, k sources

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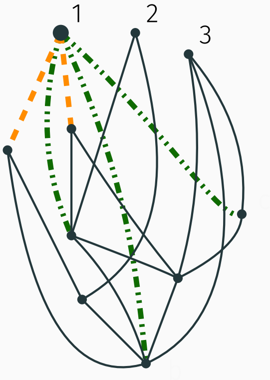

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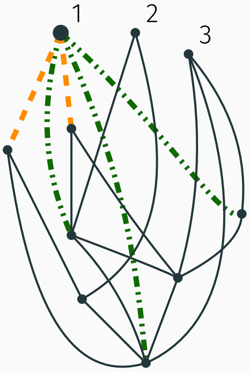

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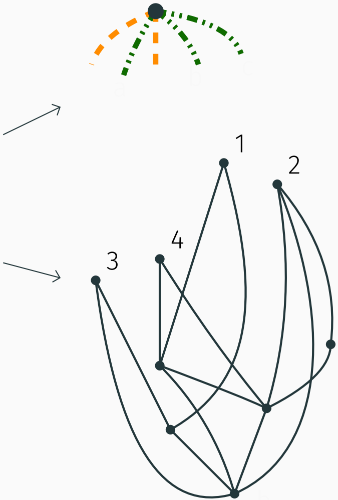

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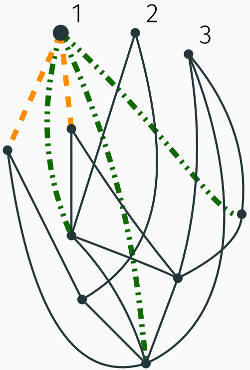

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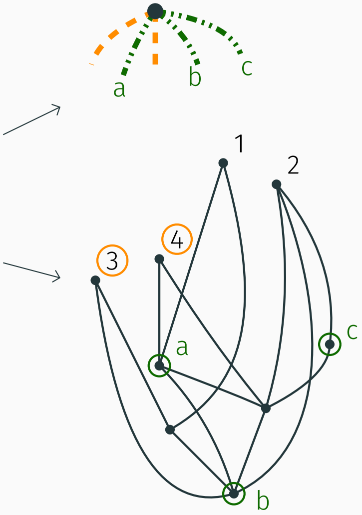

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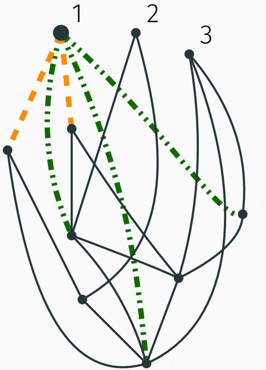

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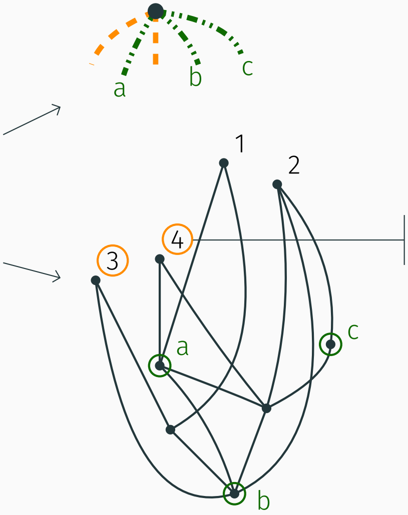

s edges to new sources

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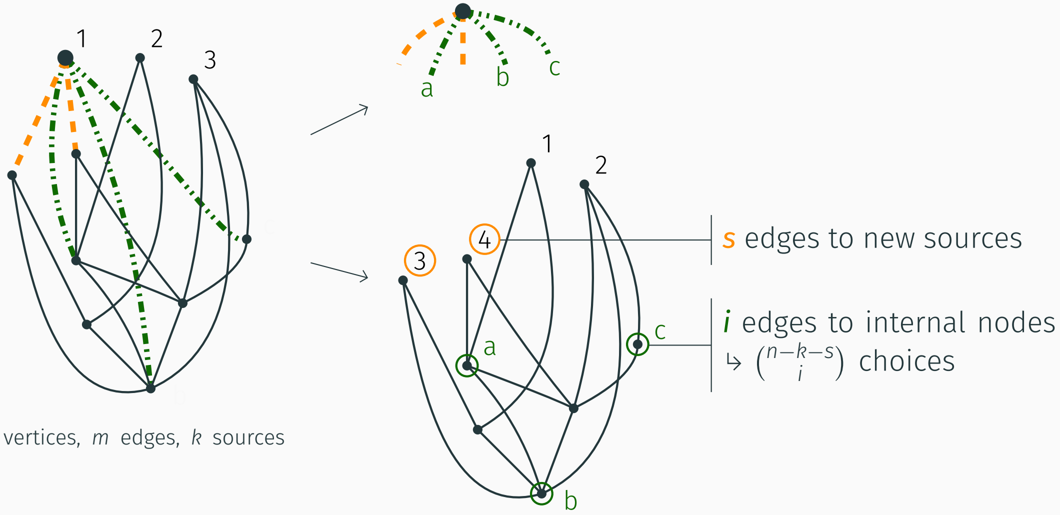

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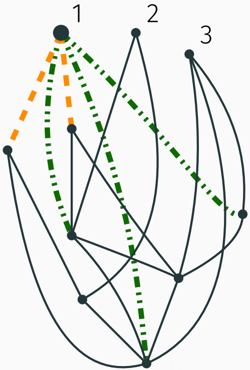

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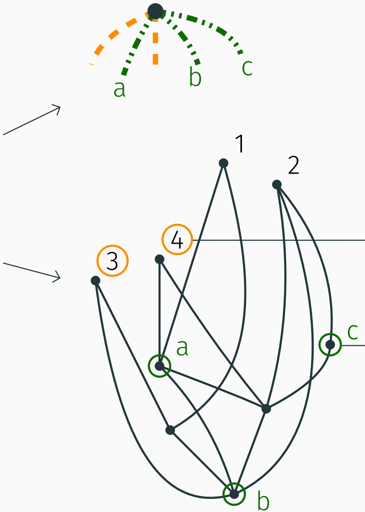

s edges to new sources

i edges to internal nodes

↳ $\binom{n-k-s}{i}$ choices

$(n-1)$ vertices, $(m-i-s)$ edges, $(k+s-1)$ sources

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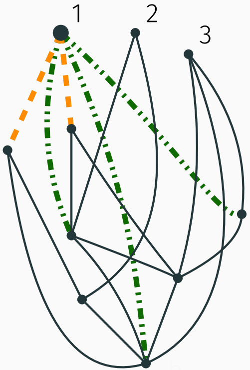

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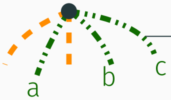
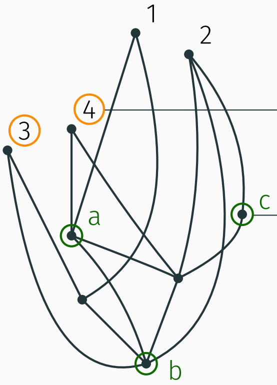


permute the […] edges
interleave […] with […]
↳ $i!\binom{i+s}{i}$ choices



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Recurrence formula

Counting formula

$$D_{n,m,k} = \#\{\text{DOAGs with } n \text{ vertices, } m \text{ edges and } k \text{ sources}\}$$
$$= \sum_{i,s \geq 0} D_{n-1,m-i-s,k+s-1} \binom{n-k-s}{i} i! \binom{i+s}{i}$$

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Counting: up to $n, k = N$ and $m = M$:

→ $O(N^4 M)$ operations;

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In practice: for $M \approx 400$, one sink: → counting = several minutes
→ sampling = a few ms

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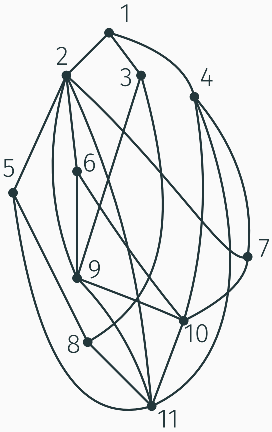
Number of single-source DOAGs (P., Viola, 2023+)

$$D_n \underset{n\to\infty}{\sim} c \cdot n^{-1/2} \cdot e^{n-1} \cdot ¡n-1!$$

for $c \approx 0.4967$ *and where* $¡x! = \prod_{k=1}^{x} k!$.

Matrix encoding



1	2	3	4	5	6	7	8	9	10	11	
											1
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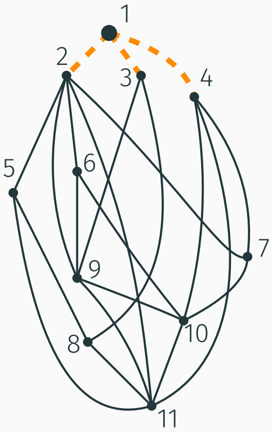

	1	2	3	4	5	6	7	8	9	10	11
1		**1**	**2**	**3**							
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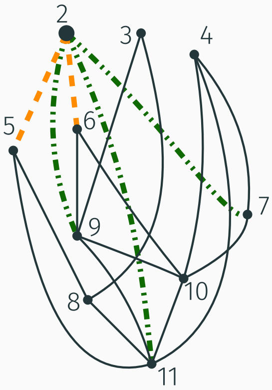

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				1	**3**	**5**		**2**		**4**	2
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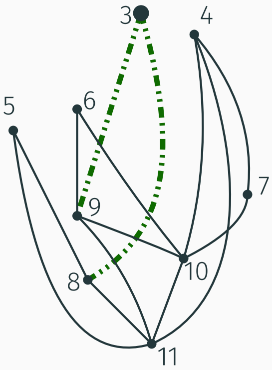

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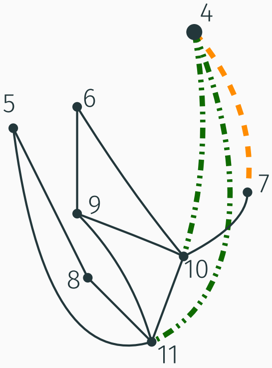

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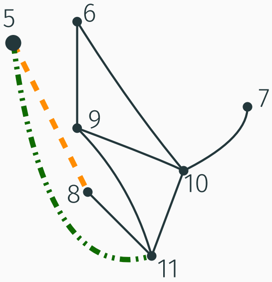

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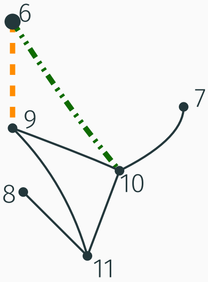

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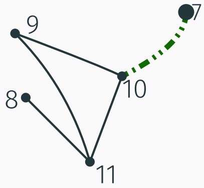

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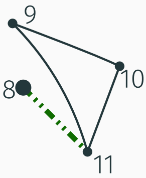

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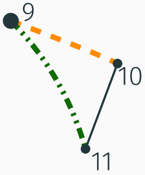

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Matrix encoding

1. strict upper triangular matrix;

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2. lines use an interval of values;

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2. lines use an interval of values;
3. $a_{1,2} \neq 0$;

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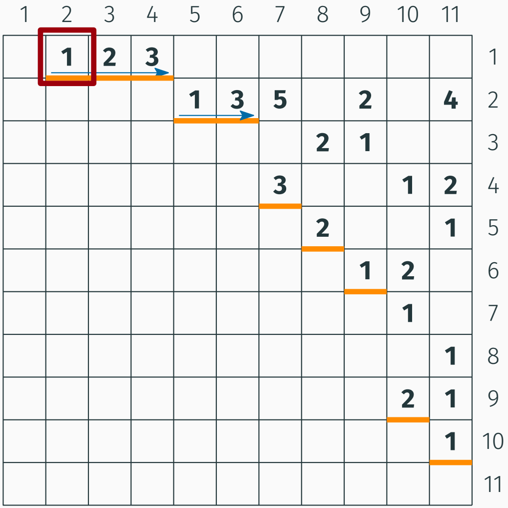
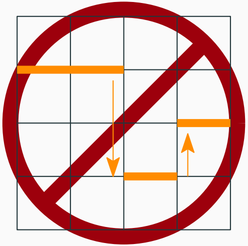
											11

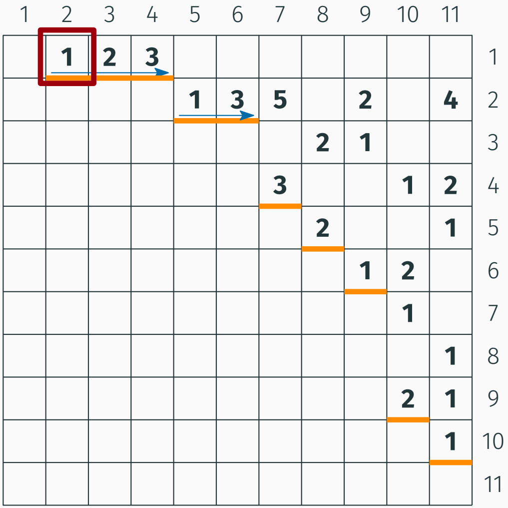
1. strict upper triangular matrix;
2. lines use an interval of values;
3. $a_{1,2} \neq 0$;
4. increasing numbers above orange lines;



Matrix encoding

1. strict upper triangular matrix;
2. lines use an interval of values;
3. $a_{1,2} \neq 0$;
4. increasing numbers above orange lines;
5. orange lines go down.





Proof sketch (1/3)

The plan: 1. Upper bound 2. Lower bound 3. Bootstrapping

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6	**1**		**5**		**2**	**4**		**3**	$=$	**6**	**1**	**5**	**2**	**4**	**3**	$\sqcup\!\sqcup$			

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$$\text{Variation} = \textsc{Seq}(\mathcal{Z}) \star \textsc{Set}(\mathcal{Z})$$

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$=$

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$$V(z) = (1-z)^{-1} e^{z}$$

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6	1		5		2	4		3	=	6	1	5	2	4	3	⧢				

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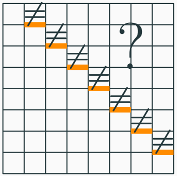
$$D_n \leq \prod_{k=1}^{n-1} v_k \leq e^{n-1}\ ¡n-1!$$

Proof sketch (2/3)

The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

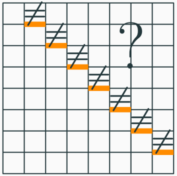
$\{$DOAG matrices$\} \supseteq$



(constraints are automatically satisfied)

Proof sketch (2/3)

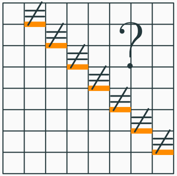
The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

{DOAG matrices} ⊇  (constraints are automatically satisfied)

$\#\,\boxed{\neq}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?} = V_k - V_{k-1}$

Proof sketch (2/3)

The plan: 1. Upper bound **2. Lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$ 

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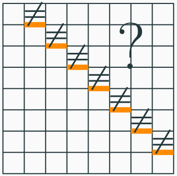
$$\#\,\boxed{\neq}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?}\boxed{?} = v_k - v_{k-1} = e \cdot k! \cdot \left(1 - \frac{1}{k} - o\left(\frac{1}{(k-1)!}\right)\right)$$

Proof sketch (2/3)

The plan:	1. Upper bound	**2. Lower bound**	3. Bootstrapping

{DOAG matrices} $\supseteq$  (constraints are automatically satisfied)

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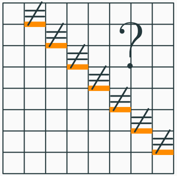
{DOAG matrices} $\supseteq$  (constraints are automatically satisfied)

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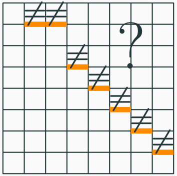
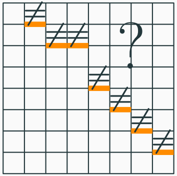
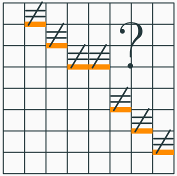
$$D_n \geq e^{n-1}\ ¡n-1! \prod_{k=2}^{n-1}\left(\frac{k-1}{k} + o\left(\frac{1}{(k-1)!}\right)\right) \geq e^{n-1}\ ¡n-1!\ \frac{A}{n} \quad \text{for some } A > 0$$

Proof sketch (2'/3)

The plan: 1. Upper bound **2'. Better lower bound** 3. Bootstrapping

{DOAG matrices} $\supseteq$  $+$  $+$  $+ \cdots$

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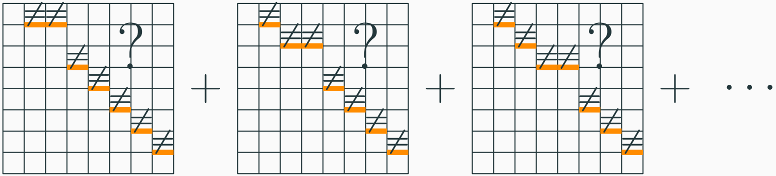
{DOAG matrices} ⊇ 

$$D_n \geq \frac{A' \cdot \ln(n)}{n} e^{n-1} \text{¡} n - 1 \text{!}$$

Proof sketch (2'/3)

The plan: 1. Upper bound **2'. Better lower bound** 3. Bootstrapping

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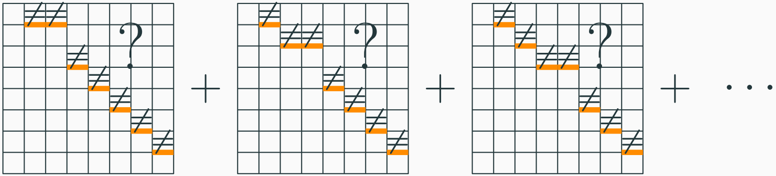
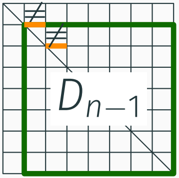


$$D_n \geq \frac{A' \cdot \ln(n)}{n}\, e^{n-1}\, \text{¡}n - 1!$$

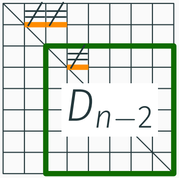
$$P_n = \frac{D_n}{e^{n-1}\, \text{¡}n - 1!} \quad \Rightarrow \quad \frac{A' \cdot \ln(n)}{n} \leq P_n \leq 1$$

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

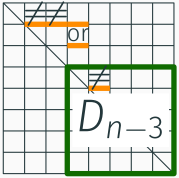
{DOAG matrices} =



+



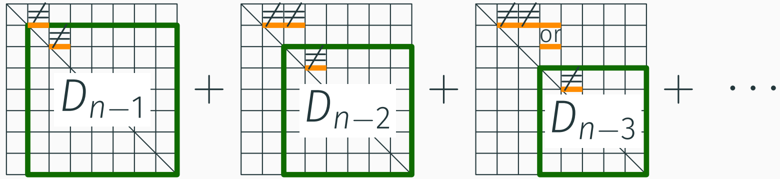
+



+ $\cdots$

Proof sketch (3/3)

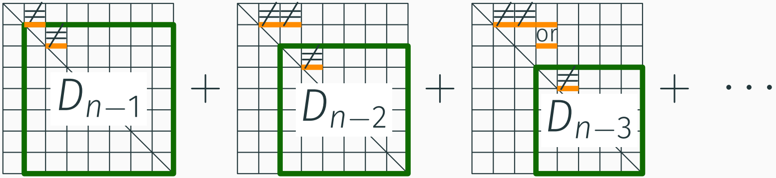
The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

{DOAG matrices} = 

$$D_n = (v_{n-1} - v_{n-2})D_{n-1} + \frac{1}{2}(v_{n-1} - 2v_{n-2} + v_{n-3})v_{n-3}D_{n-2} + \cdots$$

Proof sketch (3/3)

The plan: 1. Upper bound 2. Better lower bound **3. Bootstrapping**

{DOAG matrices} = 

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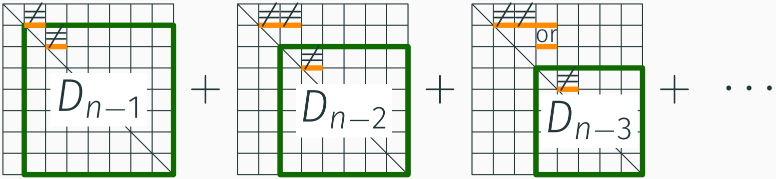

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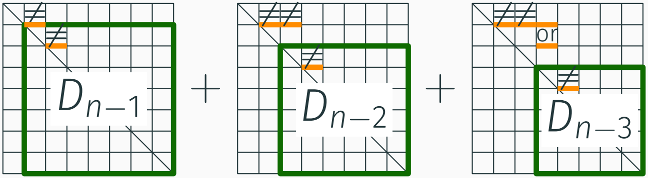

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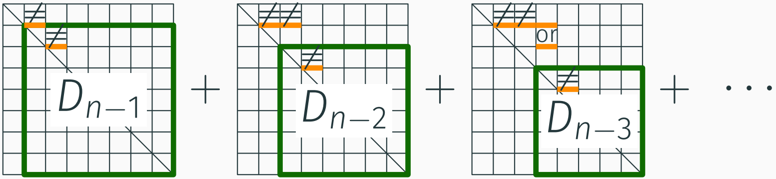

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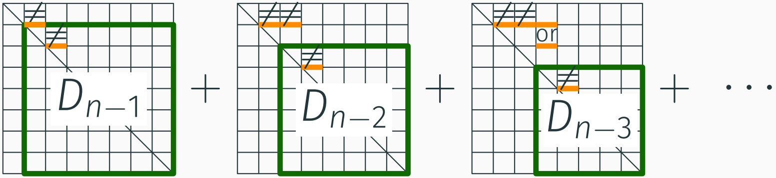

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$$P_n = P_{n-1}\left(1 - \frac{1}{2n} + O\left(n^{-2}\right)\right) \Rightarrow \boxed{P_n \sim c \cdot n^{-1/2}}$$

Random sampling again!

Corollary

$$\frac{D_n}{\#\{\text{matrices of variations of sizes } 1, 2, \ldots, n-1\}} \sim c \cdot n^{-\frac{1}{2}}$$

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Generating one variation: $\sim n \log_2(n)$ random bits.

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    - Simpler recurrence relation
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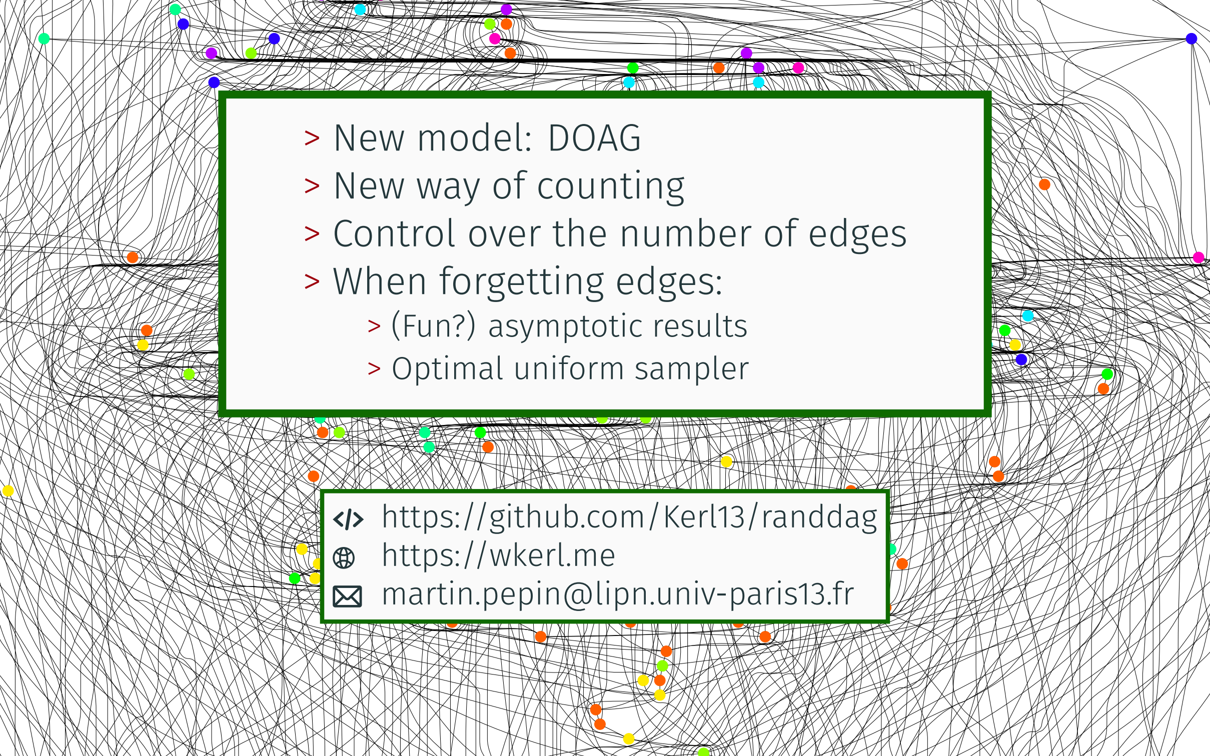
- What about sparse DOAGs?



- New model: DOAG
- New way of counting
- Control over the number of edges
- When forgetting edges:
  - (Fun?) asymptotic results
  - Optimal uniform sampler

https://github.com/Kerl13/randdag
https://wkerl.me
martin.pepin@lipn.univ-paris13.fr

Outline of the presentation

Background

Directed ordered acyclic graphs

↳ *definition and recursive decomposition*

Asymptotic analysis

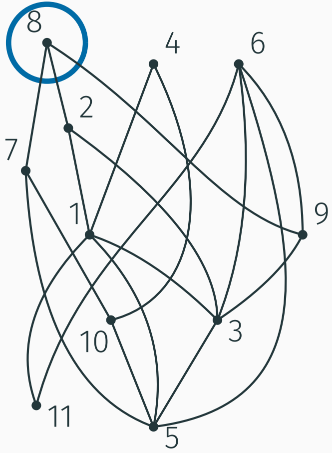
↳ *matrix encoding*
↳ *asymptotic result*
↳ *faster sampler*

Bonus: labelled DAGs

↳ *another way of counting*

What about labelled DAGs?

Idea: mark one source, and remove it.



$A_{n,m,k} = \#\text{DAGs}\ (n \text{ vertices}, m \text{ edges}, k \text{ sources})$

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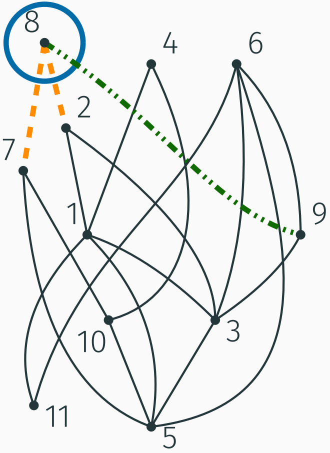

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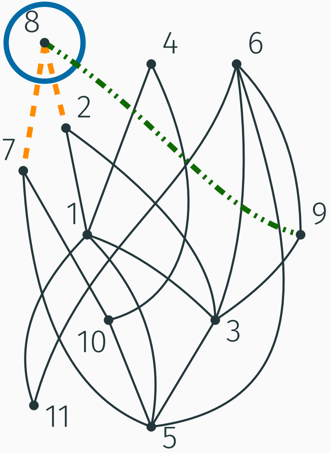

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→ New counting formula for DAGs;

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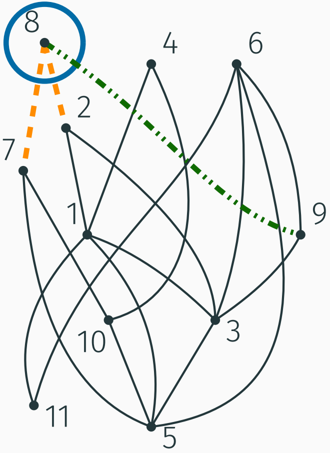

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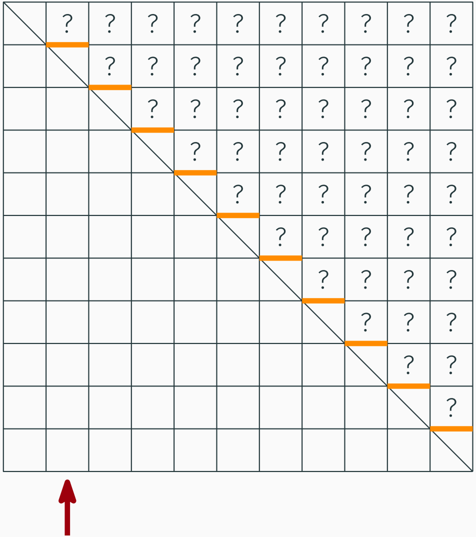
→ New counting formula for DAGs;

→ Effective sampler with fixed number of edges and vertices.

Anticipated rejection



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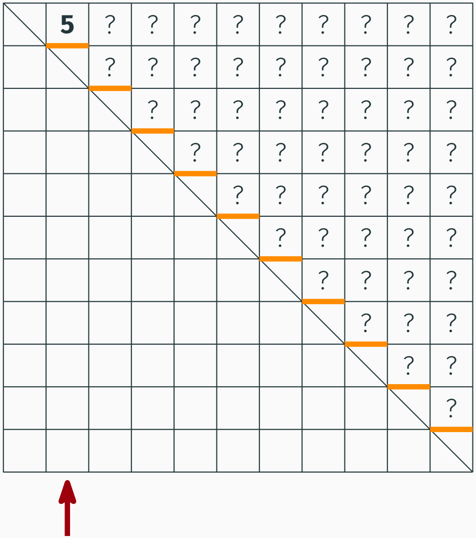

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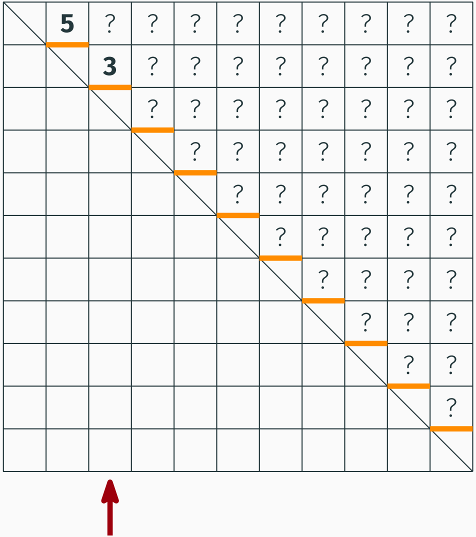
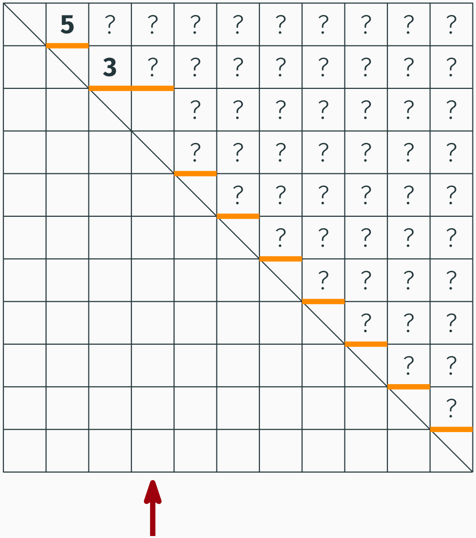


Anticipated rejection



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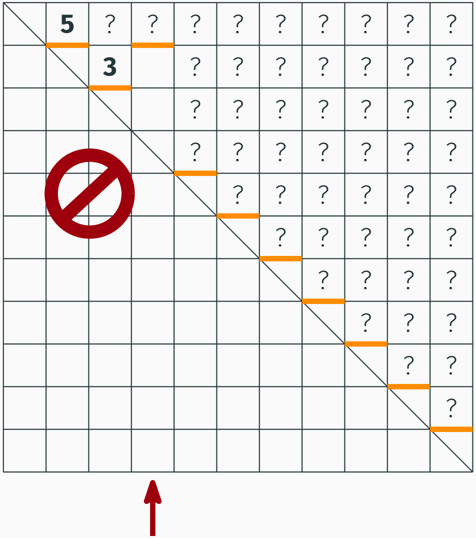

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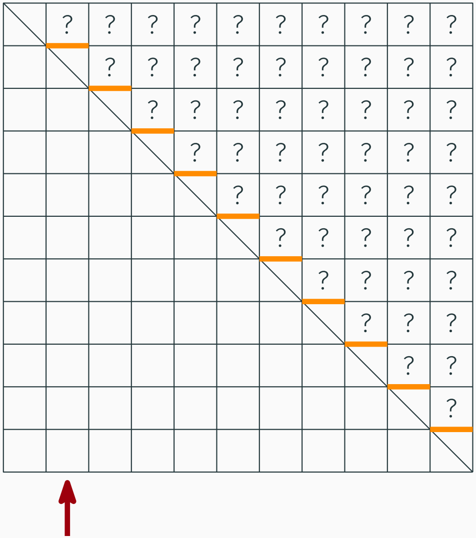

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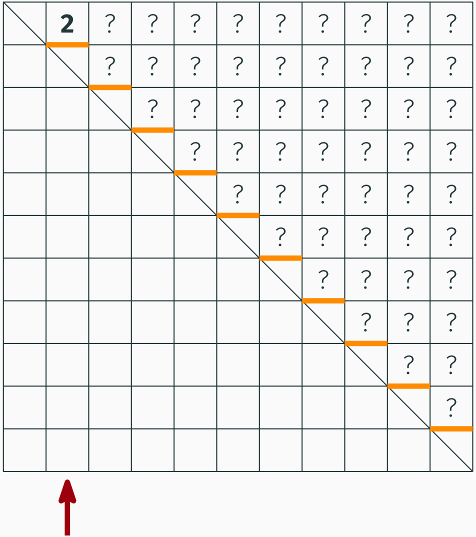

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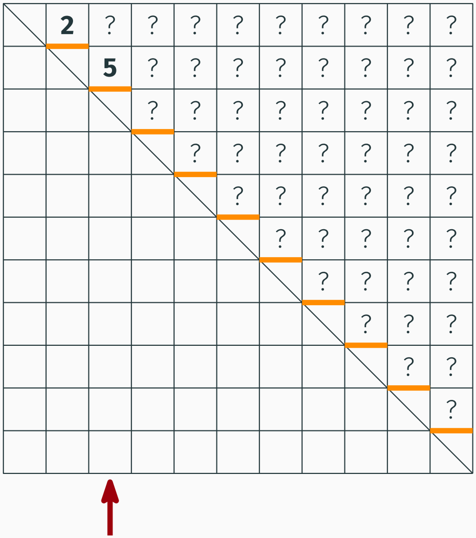

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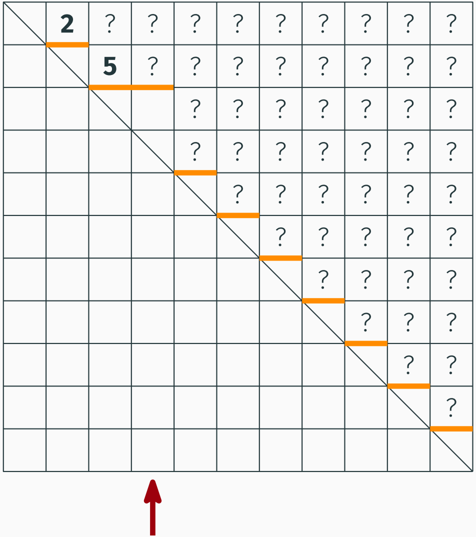

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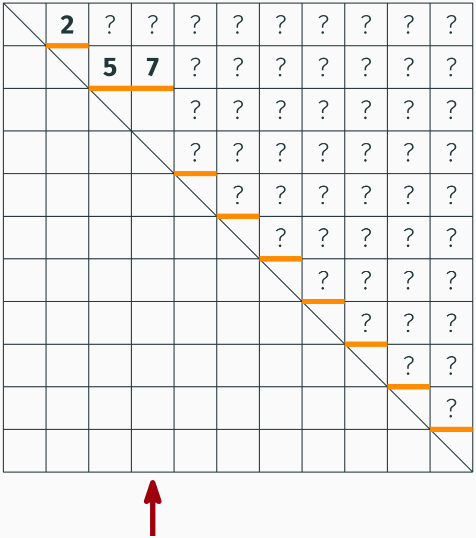

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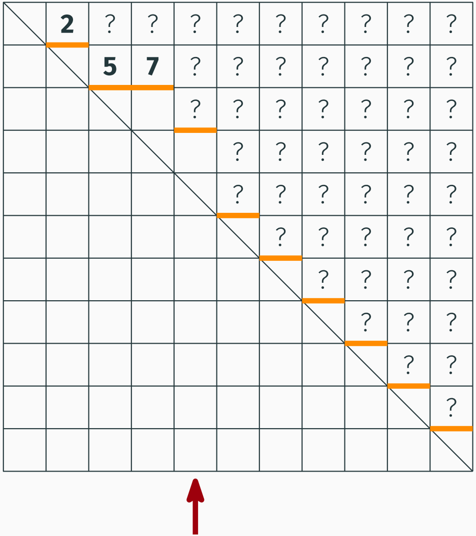

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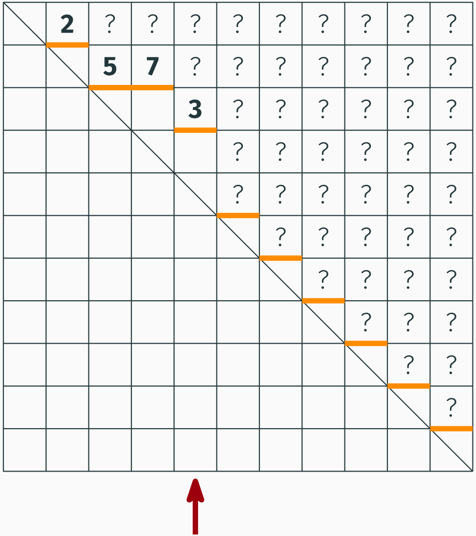

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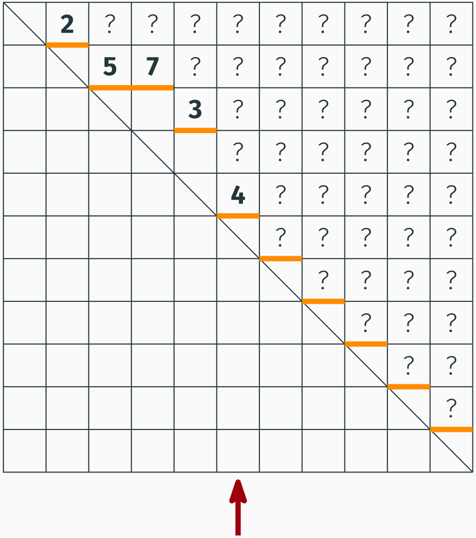

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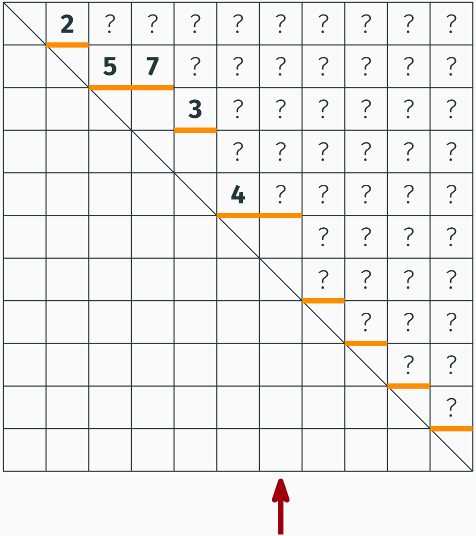

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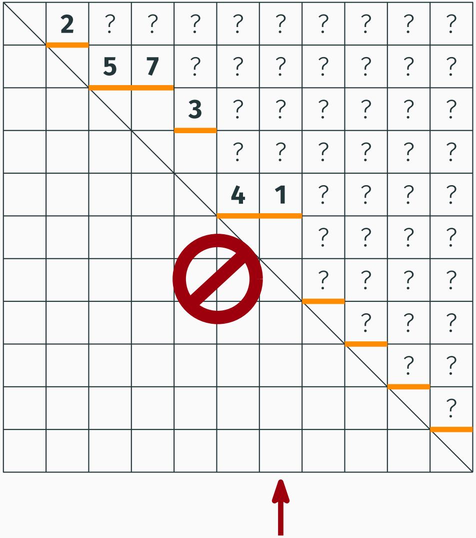

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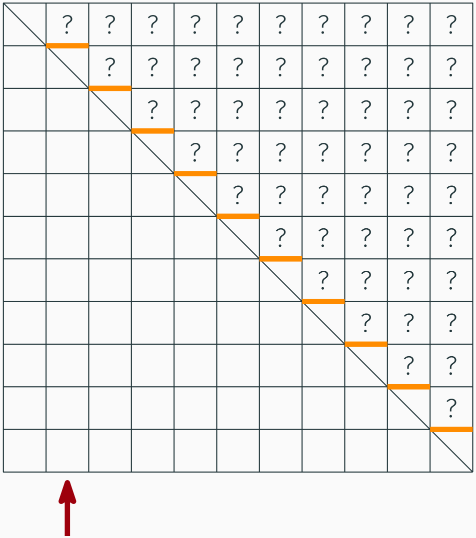

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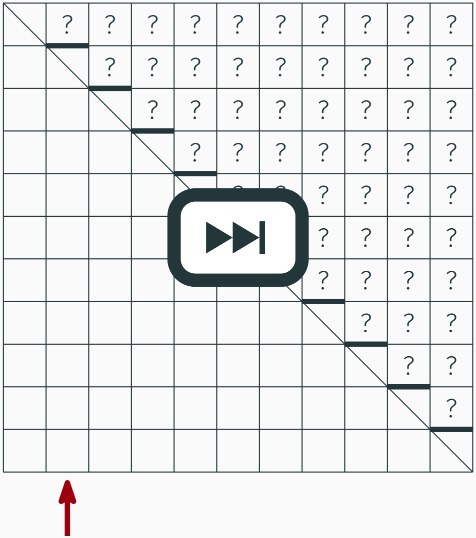

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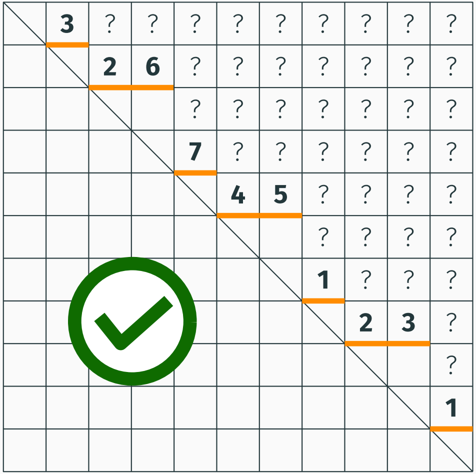

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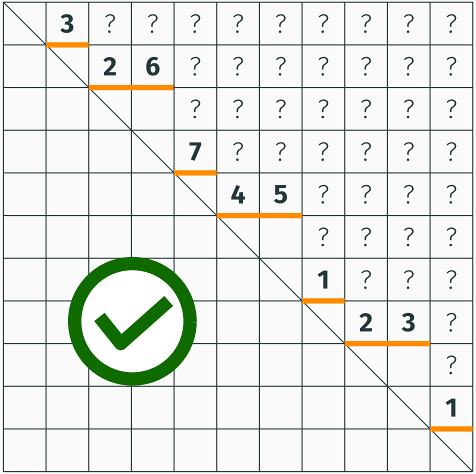

Complexity $= O(n \ln(n))$

Total complexity $= \frac{n^2}{2} \log_2(n) + O\left(\sqrt{n} \cdot n \ln(n)\right)$

*1

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